Assignment 4

Arfken 5.1.2

We have the sum

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Note that the first 4 partial sums are

$$s_1 = \frac{1}{2}, \quad s_2 = \frac{2}{3}, \quad s_3 = \frac{3}{4}, \quad s_4 = \frac{4}{5}$$

so we guess that $s_n = n/(n+1)$. Proving this by induction, we see it is true for $n = 1$, we assume it is true for n and verify it for $n + 1$

$$\begin{aligned} s_{n+1} &= \sum_{i=1}^{n+1} \frac{1}{i(i+1)} \\ &= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\ &= \frac{n(n+2)+1}{(n+1)(n+2)} \\ &= \frac{n+1}{n+2} \end{aligned}$$

which completes the proof. Now, as $n \to \infty$, $s_n \to 1$ and we conclude that the sum of our infinite series is 1.

Arfken 5.2.7

(a)

$$\begin{aligned} \frac{u_n}{u_{n+1}} &= \frac{(n+1)(n+2)}{n(n+1)} \\ &= 1 + \frac{2}{n} \end{aligned}$$

which, by Gauss' test converges ($h > 1$).

(b) Use the integral test:

$$\int \frac{dx}{x \ln x} = \ln|\ln x| \Big|_{x=2}^{\infty}$$

which diverges. Therefore $\sum \frac{1}{n \ln n}$ also diverges.

(c) By the ratio test:

$$\begin{aligned} \frac{u_{n+1}}{u_n} &= \frac{n 2^n}{(n+1)2^{n+1}} \\ &= \frac{1}{2} \frac{n}{n+1} \end{aligned}$$

which goes to 1/2 as $n \to \infty$ and thus the series converges.

(d) Use the integral test:

$$\begin{aligned} \int_1^{\infty} \ln\left|1 + \frac{1}{x}\right| dx &= \int_1^{\infty} \left[\ln(1+x) - \ln x\right] dx \\ &= (x+1)\ln|x+1| - (x+1) - x\ln x + x \Big|_1^{\infty} \\ &= \ln(x+1) + x\ln(1+\frac{1}{x}) - 1 \Big|_1^{\infty} \end{aligned}$$

where the last two terms $\to 0$ at the upper limit, but notice that the first term in the last line diverges. Thus the integral and the sum diverge.

(e) Use comparison with the sum in (b). If

$$u_n = \frac{1}{n\, n^{1/n}} > \frac{1}{n \ln n}$$

then the sum $\sum u_n$ will diverge. Simplifying the inequality, we only need to show (consider $n > 1$)

$$\ln n > n^{1/n}$$

As $n \to \infty$, $\ln n \to \infty$ but

$$n^{1/n} = e^{\frac{1}{n}\ln n}$$

and the exponent goes to 0 (use, *e.g.* l'Hospital's rule), the exponential goes to 1 and the sum diverges.

Arfken 5.2.9

The hypergeometric series has as its n^{th} term

$$u_n = \frac{x^n}{n!}\frac{(\alpha+n-1)!(\beta+n-1)!(\gamma-1)!}{(\alpha-1)!(\beta-1)!(\gamma+n-1)!}$$

To find the range of convergence, use the ratio test initially

$$\begin{aligned}\frac{u_{n+1}}{u_n} &= \frac{x^{n+1}}{x^n}\frac{1}{n+1}\frac{(\alpha+n)(\beta+n)}{\gamma+n} \\ &= x\,\frac{n^2+n(\alpha+\beta)+\alpha\beta}{n^2+n(\gamma+1)+\gamma}\end{aligned}$$

Note, that for $|x| > 1$, this diverges by the ratio test while for $|x| < 1$, by the same test, it converges as $n \to \infty$. For $x = 1$, we must use a more sensitive test such as the Gauss test. For it we need the inverse of this

$$\frac{u_n}{u_{n+1}} = \frac{1}{x}\frac{n^2+n(\gamma+1)+\gamma}{n^2+n(\alpha+\beta)+\alpha\beta}$$

which is convergent for $|x| = 1$ provided $\gamma + 1 > \alpha + \beta + 1$.

Arfken 5.2.19

Anticipating using Gauss' test, we construct

$$\begin{aligned}\frac{u_n}{u_{n+1}} &= \frac{(2s-1)!!}{(2s)!!(2s+1)}\frac{(2s+2)!!(2s+3)}{(2s+1)!!} \\ &= \frac{(2s+2)(2s+3)}{(2s+1)(2s+1)} \\ &= \frac{s^2+\frac{5}{2}s+\frac{3}{2}}{s^2+s+\frac{1}{4}}\end{aligned}$$

Since $5/2 > 1 + 1$, Gauss' test is satisfied for convergence.

Arfken 5.3.1

(a) The series is

$$\sum_{s=0}^{\infty}(-1)^s(4s+3)\frac{(2s-1)!!}{(2s+2)!!}$$

Testing for absolute convergence, construct

$$\left|\frac{u_n}{u_{n+1}}\right| = \frac{4s+3}{4s+7}\,\frac{(2s-1)!!\,(2s+4)!!}{(2s+1)!!\,(2s+2)!!}$$
$$= \frac{4s+3}{4s+7}\,\frac{2s+4}{2s+1}$$
$$= \frac{s^2+11s/4+3/2}{s^2+9s/4+7/8}$$

Since $11/4 < 9/4+1$, Gauss' test tells us that this series diverges.

However, note that as an alternating series it may converge conditionally if we can show that the terms in the series are monotonically decreasing. To this end, consider the following infinite product definition of π (Eq. 5.214)

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\left[\frac{(2n)^2}{(2n+1)(2n-1)}\right]$$
$$= \lim_{n\to\infty}\left[\frac{(2n)!!}{(2n-1)!!}\right]^2 \frac{1}{2n+1}$$

which we can interpret in the following way

$$\lim_{n\to\infty}\frac{(2n)!!}{(2n-1)!!} = \lim_{n\to\infty}\sqrt{\frac{\pi}{2}(2n+1)}$$

Using this, we have

$$\lim_{s\to\infty}(4s+3)\frac{(2s-1)!!}{(2s+2)!!} = \lim_{s\to\infty}\frac{(4s+3)}{(2s)(2s+2)}\sqrt{\frac{\pi}{2}2s}$$
$$= 0$$

Thus terms are monotonically decreasing and the series satisfies the Leibniz criteria for a conditionally convergent series.

(b) The series is

$$\sum_{s=0}^{\infty}(-1)^s(4s+3)\frac{(2s-1)!!}{(2s)!!}$$

Testing for absolute convergence, construct

$$\left|\frac{u_n}{u_{n+1}}\right| = \frac{4s+3}{4s+7}\,\frac{(2s-1)!!\,(2s+2)!!}{(2s+1)!!\,(2s)!!}$$
$$= \frac{4s+3}{4s+7}\,\frac{2s+2}{2s+1}$$
$$= \frac{s^2+7s/4+3/4}{s^2+9s/4+7/8}$$

Since $7/4 < 9/4+1$, Gauss' test tells us that this series diverges.

However, note that as an alternating series it may converge conditionally if we can show that the terms in the series are monotonically decreasing. Using the argument from part (a), we have

$$\lim_{s\to\infty}(4s+3)\frac{(2s-1)!!}{(2s)!!} = \lim_{s\to\infty}\frac{(4s+3)}{2s}\sqrt{\frac{\pi}{2}2s}$$
$$\to \infty$$

Thus it does *not* satisfy the Leibniz criteria for a conditionally convergent series and hence diverges.

Arfken 5.4.3

$$\begin{aligned}
\sum_{n=2}^{\infty}[\zeta(n)-1] &= \sum_{n=2}^{\infty}\left[\sum_{k=1}^{\infty}\frac{1}{k^n}-1\right] \\
&= \sum_{n=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{k^n} \\
&= \sum_{k=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{k^n} \\
&= \sum_{k=2}^{\infty}\left(\frac{1}{k^2}+\frac{1}{k^3}+\frac{1}{k^4}+\cdots\right) \\
&= \sum_{k=2}^{\infty}\frac{1}{k^2}\frac{1}{1-\frac{1}{k}} \\
&= \sum_{k=2}^{\infty}\frac{1}{k(k-1)} \\
&= \sum_{l=1}^{\infty}\frac{1}{(l+1)l} \\
&= 1
\end{aligned}$$

where the last line uses the result of problem 5.1.2. Part (b) is virtually the same:

$$\begin{aligned}
\sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1] &= \sum_{n=2}^{\infty}(-1)^n\left[\sum_{k=1}^{\infty}\frac{1}{k^n}-1\right] \\
&= \sum_{n=2}^{\infty}\sum_{k=2}^{\infty}\left(-\frac{1}{k}\right)^n \\
&= \sum_{k=2}^{\infty}\sum_{n=2}^{\infty}\left(-\frac{1}{k}\right)^n \\
&= \sum_{k=2}^{\infty}\frac{1}{k^2}\frac{1}{1+\frac{1}{k}} \\
&= \sum_{k=2}^{\infty}\frac{1}{k(k+1)} \\
&= -\frac{1}{2}+\sum_{k=1}^{\infty}\frac{1}{k(k+1)} \\
&= \frac{1}{2}
\end{aligned}$$

Arfken 5.5.3

For the series, $\sum_{n=0}^{\infty} 1/(1+x^n)$, decide for what range of positive x values, it converges and is uniformly convergent. Use the ratio test,

$$\frac{u_{n+1}}{u_n} = \frac{1+x^n}{1+x^{n+1}}$$

If $x > 1$, this goes to $1/x < 1$ as $n \to \infty$. On the other hand, if $x \leq 1$, this ratio will go to 1 as $n \to \infty$. So we conclude that the series is absolutely convergent for $x > 1$.

For uniform convergence, we must compare this to a series of numbers, $\sum M_n$, which is convergent and for which $M_n \geq u_n$. Try the following

$$\begin{aligned} p &\leq x \\ p^n &< 1 + x^n \\ \frac{1}{1+x^n} &< \frac{1}{p^n} \\ \sum_{n=1}^{\infty} \frac{1}{1+x^n} &< \sum_{n=1}^{\infty} \frac{1}{p^n} \end{aligned}$$

Since the series of numbers converges for $p > 1$ (it's just a geometric series), our series of functions, converges uniformly for all $1 < p \leq x < \infty$.

Arfken 5.6.11

$$\begin{aligned} (1+x)^{-m/2} &= 1 + \big(-\frac{m}{2}\big)x + \big(-\frac{m}{2}\big)\big(-\frac{m}{2}-1\big)\frac{x^2}{2!} + \big(-\frac{m}{2}\big)\big(-\frac{m}{2}-1\big)\big(-\frac{m}{2}-2\big)\frac{x^3}{3!} \\ &\quad + \big(-\frac{m}{2}\big)\big(-\frac{m}{2}-1\big)\big(-\frac{m}{2}-2\big)\big(-\frac{m}{2}-3\big)\frac{x^4}{4!} + \cdots \\ &\quad + \big(-\frac{m}{2}\big)\big(-\frac{m}{2}-1\big)\cdots\big(-\frac{m}{2}-(n-2)\big)\big(-\frac{m}{2}-(n-1)\big)\frac{x^n}{n!} + \cdots \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n!}\frac{1}{2^n} m(m+2)(m+4)\cdots(m+2n-4)(m+2n-2) \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n!}\frac{1}{2^n}\frac{(m+2n-2)!!}{(m-2)!!} \end{aligned}$$

Arfken 5.6.12

The Doppler shift formulas are

$$\begin{aligned} \nu'_{(a)} &= \nu\Big[1 \pm \frac{v}{c} + \frac{v^2}{c^2} \pm \cdots\Big] \\ \nu'_{(b)} &= \nu\Big[1 \pm \frac{v}{c}\Big] \\ \nu'_{(c)} &= \nu\Big[1 \pm \frac{v}{c} + \frac{v^2}{2c^2} \pm \cdots\Big] \end{aligned}$$

Arfken 5.6.18 Two binomial expansions:

$$\frac{x}{1-x} = \sum_{n=1}^{\infty} x^n$$
$$\frac{x}{x-1} = \sum_{n=0}^{\infty} x^{-n}$$

are added together to give

$$0 = \sum_{n=-\infty}^{\infty} x^n$$

which is obviously false.

The problem comes in the ranges over which the expansions are convergent. The first expansion converges for $|x| < 1$ while the second converges for $|x| > 1$. Thus, to add the resulting expansions together, defined as they are over different ranges, makes no sense; indeed it is an undefined operation.

Arfken 5.6.22

The linear combination is

$$\begin{aligned}
& y(x-2h) - 8y(x-h) \\
& +8y(x+h) - y(x+2h) \\
&= -\Big[y(x) + 2hy'(x) + \frac{(2h)^2}{2!}y''(x) + \frac{(2h)^3}{3!}y'''(x) + \frac{(2h)^4}{4!}y^{(4)}(x) + \frac{(2h)^5}{5!}y^{(5)}(x) + \cdots\Big] \\
&\quad + \Big[y(x) - 2hy'(x) + \frac{(2h)^2}{2!}y''(x) - \frac{(2h)^3}{3!}y'''(x) + \frac{(2h)^4}{4!}y^{(4)}(x) - \frac{(2h)^5}{5!}y^{(5)}(x) + \cdots\Big] \\
&\quad + 8\Big[y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \frac{h^4}{4!}y^{(4)}(x) + \frac{h^5}{5!}y^{(5)}(x) + \cdots\Big] \\
&\quad - 8\Big[y(x) - hy'(x) + \frac{h^2}{2!}y''(x) - \frac{h^3}{3!}y'''(x) + \frac{h^4}{4!}y^{(4)}(x) - \frac{h^5}{5!}y^{(5)}(x) + \cdots\Big] \\
&= -2\Big[2hy'(x) + \frac{(2h)^3}{3!}y'''(x) + \frac{(2h)^5}{5!}y^{(5)}(x) + \cdots\Big] \\
&\quad + 16\Big[hy'(x) + \frac{(h)^3}{3!}y'''(x) + \frac{(h)^5}{5!}y^{(5)}(x) + \cdots\Big] \\
&= \qquad 12h\,y'(x) - \frac{48\,h^5}{5!}\,y^{(5)}(x) + \cdots
\end{aligned}$$

so that dividing by $12h$ gives the answer in the text.