Assignment 5

Arfken 5.7.2

We have the quantity L in oblate spheroidal coordinates

$$L = \frac{1}{\epsilon_0}(1+\zeta_0^2)(1-\zeta_0\,\cot^{-1}\zeta_0)$$

We want the limits of L as the parameter $\zeta_0 \to \infty$ and $\zeta_0 \to 0$. To this end we need an expansion for $\cot^{-1}\zeta_0$ for both small and large ζ_0. The first of these is

$$\cot^{-1} x = \frac{\pi}{2} - x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots$$

which is valid for $x^2 < 1$. To get the expansion for large ζ_0, we need to use the identity $\cot^{-1} x = \tan^{-1}(1/x)$. This allows us to use the expansion for $\tan^{-1}()$ for small values of the argument to get the expansion for $\cot^{-1}()$ for large values of its argument:

$$\cot^{-1} x = \tan^{-1}\frac{1}{x} = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \cdots$$

which is valid for $1/x^2 < 1$, *i.e.* $x^2 > 1$. Using now the appropriate expansions, we find

$$\begin{aligned}
\lim_{\zeta_0\to\infty} L &= \frac{1}{\epsilon_0}(1+\zeta_0^2)\big(1-\zeta_0\,\left[\frac{1}{\zeta_0} - \frac{1}{3\zeta_0^3} + \frac{1}{5\zeta_0^5} - \cdots\right]\big) \\
&= \frac{1}{\epsilon_0}(1+\zeta_0^2)\frac{1}{3\zeta_0^2} + \cdots \\
&= \frac{1}{3\epsilon_0} \\
\lim_{\zeta_0\to 0} L &= \frac{1}{\epsilon_0}(1+\zeta_0^2)\big(1-\zeta_0\,\left[\frac{\pi}{2} - \zeta_0 + \frac{\zeta_0^3}{3} - \cdots\right]\big) \\
&= \frac{1}{\epsilon_0}
\end{aligned}$$

Arfken 5.7.7

$$\begin{aligned}
\int_0^x e^{-t} t^n\,dt &= \int_0^x t^n \sum_{p=0}^{\infty}\frac{(-t)^p}{p!}\,dt \\
&= \sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\int_0^x t^{n+p}\,dt \\
&= \sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\frac{x^{n+p+1}}{n+p+1}
\end{aligned}$$

The radius of convergence is most easily determined via the ratio test

$$\begin{aligned}
\lim_{p\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{p\to\infty}\frac{x^{n+p+2}}{(n+p+2)(p+1)!}\frac{p!(n+p+1)}{x^{n+p+1}} \\
&= \lim_{p\to\infty} x\frac{n+p+1}{(n+p+2)(p+1)} \\
&= 0
\end{aligned}$$

for all values of x. Thus this converges for all x.

Arfken 5.7.15

We want the limit as $\epsilon \to 0$ for $f(\epsilon)$

$$\begin{aligned}
\lim_{\epsilon\to 0} f(\epsilon) &= \lim_{\epsilon\to 0} \frac{(1+\epsilon)}{\epsilon^2} \left[\frac{2+2\epsilon}{1+2\epsilon} - \frac{\ln(1+2\epsilon)}{\epsilon}\right] \\
&= \lim_{\epsilon\to 0} \frac{(1+\epsilon)}{\epsilon^2} \left[(2+2\epsilon)\left(1-2\epsilon+4\epsilon^2-8\epsilon^3+\cdots\right) - \frac{1}{\epsilon}\left(2\epsilon - \frac{4\epsilon^2}{2} + \frac{8\epsilon^3}{3} - \cdots\right)\right] \\
&= \lim_{\epsilon\to 0} \frac{(1+\epsilon)}{\epsilon^2} \left[2-2\epsilon+4\epsilon^2-8\epsilon^3+\cdots-2+2\epsilon-\frac{8\epsilon^2}{3}+\cdots\right] \\
&= 4 - \frac{8}{3} = \frac{4}{3}
\end{aligned}$$

Arfken 5.8.4

Dropping extra factors for the moment, the integral we need to do is

$$\begin{aligned}
\int_0^\pi \frac{\cos\alpha d\alpha}{\left(a^2+\rho^2+z^2-2a\rho\cos\alpha\right)^{1/2}} &= \int_0^\pi \frac{\cos\alpha\, d\alpha}{\left(a^2+2a\rho+\rho^2+z^2-2a\rho-2a\rho\cos\alpha\right)^{1/2}} \\
&= \frac{1}{\left((a+\rho)^2+z^2\right)^{1/2}} \int_0^\pi \frac{\cos\alpha\, d\alpha}{\left(1-\frac{4a\rho\cos^2(\alpha/2)}{(a+\rho)^2+z^2}\right)^{1/2}} \\
&= \frac{k}{\sqrt{4a\rho}} \int_0^\pi \frac{\cos\alpha\, d\alpha}{\left(1-k^2\cos^2(\alpha/2)\right)^{1/2}}
\end{aligned}$$

Now let $\theta = \pi/2 - \alpha/2$.

$$\begin{aligned}
\int_0^\pi \frac{\cos\alpha d\alpha}{\left(a^2+\rho^2+z^2-2a\rho\cos\alpha\right)^{1/2}} &= \frac{k}{\sqrt{4a\rho}} \int_{\pi/2}^0 \frac{-\cos 2\theta\,(-2d\theta)}{\left(1-k^2\sin^2\theta\right)^{1/2}} \\
&= -\frac{1}{k\sqrt{a\rho}} \int_0^{\pi/2} \frac{k^2-2k^2\sin^2\theta}{\left(1-k^2\sin^2\theta\right)^{1/2}} \\
&= -\frac{1}{k\sqrt{a\rho}} \int_0^{\pi/2} \frac{k^2-2+2(1-k^2\sin^2\theta)}{\left(1-k^2\sin^2\theta\right)^{1/2}} \\
&= -\frac{1}{k\sqrt{a\rho}} \left[(k^2-2)K(k^2)+2E(k^2)\right]
\end{aligned}$$

Including the multiplicative constants now we get

$$\begin{aligned}
A_\varphi(\rho,\varphi,z) &= \frac{a\mu_0 I}{2\pi} \int_0^\pi \frac{\cos\alpha d\alpha}{\left(a^2+\rho^2+z^2-2a\rho\cos\alpha\right)^{1/2}} \\
&= \frac{\mu_0 I}{\pi k}\left(\frac{a}{\rho}\right)^{1/2} \left[(1-\frac{k^2}{2})K(k^2)-E(k^2)\right]
\end{aligned}$$

Arfken 5.8.6

(a)

$$\begin{aligned}
\frac{dE(k^2)}{dk} &= \frac{d}{dk}\int_0^{\pi/2}\left(1-k^2\sin^2\theta\right)^{1/2}\,d\theta \\
&= \int_0^{\pi/2}\frac{-k\sin^2\theta}{\left(1-k^2\sin^2\theta\right)^{1/2}}\,d\theta \\
&= \int_0^{\pi/2}\frac{1}{k}\frac{1-k^2\sin^2\theta-1}{\left(1-k^2\sin^2\theta\right)^{1/2}}\,d\theta \\
&= \frac{1}{k}\int_0^{\pi/2}\left[\left(1-k^2\sin^2\theta\right)^{1/2}-\left(1-k^2\sin^2\theta\right)^{-1/2}\right]\,d\theta \\
&= \frac{1}{k}\left(E(k^2)-K(k^2)\right)
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{dK(k^2)}{dk} &= \frac{d}{dk}\int_0^{\pi/2}\left(1-k^2\sin^2\theta\right)^{-1/2}\,d\theta \\
&= \int_0^{\pi/2}\frac{k\sin^2\theta}{\left(1-k^2\sin^2\theta\right)^{3/2}}\,d\theta \\
&= -\int_0^{\pi/2}\frac{1}{k}\frac{1-k^2\sin^2\theta-1}{\left(1-k^2\sin^2\theta\right)^{3/2}}\,d\theta \\
&= -\frac{K(k^2)}{k}+\frac{1}{k}\int_0^{\pi/2}\left(1-k^2\sin^2\theta\right)^{-3/2}\,d\theta
\end{aligned}$$

The final integral we do with help from the hint

$$\begin{aligned}
\int_0^{\pi/2}\left(1-k^2\sin^2\theta\right)^{-3/2}\,d\theta &= \int_0^{\pi/2}\Big(1+(-\frac{3}{2})(-k^2\sin^2\theta)+(-\frac{3}{2})(-\frac{5}{2})(-k^2\sin^2\theta)^2\frac{1}{2!} \\
&\qquad +(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})(-k^2\sin^2\theta)^3\frac{1}{3!}+\cdots\Big)\,d\theta \\
&= \int_0^{\pi/2}\sum_{n=0}^{\infty}\frac{k^{2n}\sin^{2n}\theta}{n!\,2^n}\,(2n+1)!!\,d\theta \\
&= \sum_{n=0}^{\infty}\frac{k^{2n}\,(2n+1)!!}{n!\,2^n}\int_0^{\pi/2}\sin^{2n}\theta\,d\theta \\
&= \sum_{n=0}^{\infty}\frac{k^{2n}(2n+1)!!}{n!\,2^n}\,\frac{\pi}{2}\cdot\frac{(2n-1)!!}{(2n)!!} \\
&= \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2(2n+1)\,k^{2n}
\end{aligned}$$

where we have used the trig integral in the text (Eq. 5.135) in going from the third to the fourth line.

Multiplying this by $(1-k^2)$ we get two sums

$$\begin{aligned}
(1-k^2)\,\frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2(2n+1)\,k^{2n} &= \frac{\pi}{2}\left(\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2(2n+1)\,k^{2n}\right.\\
&\qquad\left.-\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2(2n+1)\,k^{2n+2}\right)\\
&= \frac{\pi}{2}\left(1+\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2(2n+1)\,k^{2n}\right.\\
&\qquad\left.-\sum_{n=1}^{\infty}\left[\frac{(2n-3)!!}{(2n-2)!!}\right]^2(2n-1)\,k^{2n}\right)\\
&= \frac{\pi}{2}\left(1+\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 k^{2n}\left[\frac{(2n+1)(2n-1)}{(2n-1)}-\frac{(2n)^2}{2n-1}\right]\right)\\
&= \frac{\pi}{2}\left(1-\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\frac{k^{2n}}{2n-1}\right)
\end{aligned}$$

which is exactly the expansion for $E(k^2)$. Putting it all together, we have

$$\frac{dK(k^2)}{dk}=-\frac{K(k^2)}{k}+\frac{1}{k}\frac{E(k^2)}{1-k^2}$$

Arfken 5.9.8

Evaluate the following

$$\begin{aligned}
\int_0^\infty \frac{x^n e^x}{(e^x-1)^2}\,dx &= \int_0^\infty \frac{x^n e^{-x}}{(1-e^{-x})^2}\,dx\\
&= \int_0^\infty x^n e^{-x}\sum_{k=0}^{\infty}(k+1)e^{-kx}\,dx\\
&= \sum_{k=0}^{\infty}(k+1)\int_0^\infty x^n e^{-(k+1)x}\,dx\\
&= \sum_{k=0}^{\infty}(k+1)\int_0^\infty \frac{u^n}{(k+1)^n}e^{-u}\,\frac{du}{k+1}\\
&= \sum_{k=0}^{\infty}\frac{1}{(k+1)^n}\,n!\\
&= \sum_{p=1}^{\infty}\frac{1}{p^n}\,n!\\
&= \zeta(n)\,n!
\end{aligned}$$

where we used a binomial expansion in the third line and the substitution $u=(k+1)x$ in the fifth line. We have been a bit careless, but note that the Riemann ζ-function is only valid for $n>1$. Otherwise, the sum and integral diverge.

Arfken 5.9.16

The Debye functions are (a)

$$\begin{aligned}
\int_0^x \frac{t^n\,dt}{e^t-1} &= \int_0^x \sum_{p=0}^{\infty} \frac{B_p t^{n+p-1}}{p!}\,dt \\
&= \sum_{p=0}^{\infty} \frac{B_p x^{n+p}}{p!\,(n+p)} \\
&= x^n \left\{ \frac{1}{n} + (-\frac{1}{2})\frac{x}{n+1} + \sum_{k=1}^{\infty} \frac{B_{2k}x^{2k}}{(2k)!\,(2k+n)} \right\}
\end{aligned}$$

and (b)

$$\begin{aligned}
\int_x^\infty \frac{t^n\,dt}{e^t-1} &= \int_x^\infty t^n e^{-t} \frac{1}{1-e^{-t}}\,dt \\
&= \int_x^\infty t^n e^{-t} \sum_{p=0}^{\infty} (e^{-t})^p\,dt \\
&= \sum_{p=0}^{\infty} \int_x^\infty t^n e^{-t(p+1)}\,dt \\
&= \sum_{k=1}^{\infty} \int_x^\infty t^n e^{-kt}\,dt
\end{aligned}$$

We now do the remaining integral by parts n times and get

$$\begin{aligned}
\int_0^x \frac{t^n\,dt}{e^t-1} &= \sum_{k=1}^{\infty} \left\{ -\frac{1}{k} t^n e^{-kt} - \frac{n}{k^2} t^{n-1} e^{-kt} - \frac{n(n-1)}{k^3} t^{n-2} e^{-kt} - \cdots - \frac{n!}{k^n} \int e^{kt} dt \right\} \Bigg|_0^x \\
&= \sum_{k=1}^{\infty} e^{-kx} \sum_{l=0}^{n} \frac{n!}{(n-l)!} \frac{x^{n-l}}{k^{l+1}}
\end{aligned}$$

Arfken 5.11.2

Write the infinite product in standard form

$$\begin{aligned}
\prod_{n=1}^{\infty} \left(\frac{1+a/n}{1+b/n} \right) &= \prod_{n=1}^{\infty} \left(1 + \frac{a/n - b/n}{1+b/n} \right) \\
&= \prod_{n=1}^{\infty} \left(1 + \frac{a-b}{n+b} \right)
\end{aligned}$$

Convergence may now be determined by the convergence of the corresponding infinite series

$$\sum_{n=1}^{\infty} \frac{a-b}{n+b}$$

Using the integral test, we see that

$$\int_1^\infty \frac{a-b}{x+b}\,dx = (a-b)\ln|x+b| \Bigg|_1^\infty$$

which will diverge unless $a = b$. Thus the infinite product likewise converges only for $a = b$.

Arfken 5.11.5

Consider the infinite product

$$\prod_{n=2}^{\infty}\left[1-\frac{2}{n(n+1)}\right] = \prod_{n=2}^{\infty}\left[\frac{n^2+n-2}{n(n+1)}\right]$$
$$= \prod_{n=2}^{\infty}\left[\frac{(n+2)(n-1)}{n(n+1)}\right]$$
$$= \prod_{k=1}^{\infty}\left[\frac{(k+3)k}{(k+1)(k+2)}\right]$$

consider now the partial product, p_n

$$p_n = \frac{(n+3)!\,n!}{3!}\,\frac{1!\,2!}{(n+1)!\,(n+2)!}$$
$$= \frac{n+3}{n+1}\,\frac{2!}{3!}$$

As $n \to \infty$, the limit of the partial product becomes the value of the infinite product. Thus in this case,

$$\prod_{n=2}^{\infty}\left[1-\frac{2}{n(n+1)}\right] = \frac{1}{3}$$