Assignment 6

Arfken 6.1.7

Do both parts together:

$$\begin{aligned}
\sum_{n=0}^{N-1} (\cos nx + i \sin nx) &= \sum_{n=0}^{N-1} e^{inx} \\
&= \sum_{n=0}^{N-1} \left(e^{ix}\right)^n \\
&= \frac{1 - e^{iNx}}{1 - e^{ix}} \\
&= \frac{e^{-iNx/2} - e^{iNx/2}}{e^{-ix/2} - e^{ix/2}} \, \frac{e^{iNx/2}}{e^{ix/2}} \\
&= \frac{-2i \sin N\big(x/2\big)}{-2i \sin\big(x/2\big)} \left(\cos\big(N-1\big)\frac{x}{2} + i \sin\big(N-1\big)\frac{x}{2}\right)
\end{aligned}$$

Taking the real and imaginary parts, we have

$$\begin{aligned}
\sum_{n=0}^{N-1} \cos nx &= \frac{\sin N\big(x/2\big)}{\sin\big(x/2\big)} \cos\big(N-1\big)\frac{x}{2} \\
\sum_{n=0}^{N-1} \sin nx &= \frac{\sin N\big(x/2\big)}{\sin\big(x/2\big)} \sin\big(N-1\big)\frac{x}{2}
\end{aligned}$$

Arfken 6.1.8

For $-1 < p < 1$, do both parts together:

$$\begin{aligned}
\sum_{n=0}^{\infty} p^n (\cos nx + i \sin nx) &= \sum_{n=0}^{\infty} p^n e^{inx} \\
&= \sum_{n=0}^{\infty} \left(pe^{ix}\right)^n \\
&= \frac{1}{1 - pe^{ix}} \\
&= \frac{1 - pe^{-ix}}{(1 - pe^{ix})(1 - pe^{-ix})} \\
&= \frac{1 - p\cos x + i\, p \sin x}{1 + p^2 - p\big(e^{ix} + e^{-ix}\big)} \\
&= \frac{1 - p\cos x + i\, p \sin x}{1 + p^2 - 2p \cos x}
\end{aligned}$$

Taking the real and imaginary parts, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} p^n \cos nx &= \frac{1 - p\cos x}{1 + p^2 - 2p\cos x} \\
\sum_{n=0}^{\infty} p^n \sin nx &= \frac{p \sin x}{1 + p^2 - 2p\cos x}
\end{aligned}$$

Arfken 6.1.16

(a) Show $\sin^{-1} z = -i\ln(iz \pm \sqrt{1-z^2})$. Take the sine of both sides and demonstrate equality

$$\begin{aligned}
z &= \sin\left(-i\ln(iz \pm \sqrt{1-z^2})\right) \\
&= \frac{1}{2i}\left(e^{i\cdot -i\ln(iz\pm\sqrt{1-z^2})} - e^{-i\cdot -i\ln(iz\pm\sqrt{1-z^2})}\right) \\
&= \frac{1}{2i}\left((iz \pm \sqrt{1-z^2}) - (iz \pm \sqrt{1-z^2})^{-1}\right) \\
&= \frac{1}{2i}\left((iz \pm \sqrt{1-z^2}) - \frac{-iz \pm \sqrt{1-z^2}}{(iz \pm \sqrt{1-z^2})(-iz \pm \sqrt{1-z^2})}\right) \\
&= \frac{1}{2i}\left((iz \pm \sqrt{1-z^2}) - (-iz \pm \sqrt{1-z^2})\right) \\
&= z
\end{aligned}$$

which demonstrates what we wanted to show.

(f) Show $\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$. Do as before:

$$\begin{aligned}
z &= \tanh\left(\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)\right) \\
&= \frac{e^{\ln\left(\frac{1+z}{1-z}\right)^{1/2}} - e^{-\ln\left(\frac{1+z}{1-z}\right)^{1/2}}}{e^{\ln\left(\frac{1+z}{1-z}\right)^{1/2}} + e^{-\ln\left(\frac{1+z}{1-z}\right)^{1/2}}} \\
&= \frac{\left(\frac{1+z}{1-z}\right)^{1/2} - \left(\frac{1-z}{1+z}\right)^{1/2}}{\left(\frac{1+z}{1-z}\right)^{1/2} + \left(\frac{1-z}{1+z}\right)^{1/2}} \\
&= \frac{\left(\frac{1+z}{1-z}\right)^{1/2} - \left(\frac{1-z}{1+z}\right)^{1/2}}{\left(\frac{1+z}{1-z}\right)^{1/2} + \left(\frac{1-z}{1+z}\right)^{1/2}} \frac{\left(\frac{1+z}{1-z}\right)^{1/2} - \left(\frac{1-z}{1+z}\right)^{1/2}}{\left(\frac{1+z}{1-z}\right)^{1/2} - \left(\frac{1-z}{1+z}\right)^{1/2}} \\
&= \frac{\left(\frac{1+z}{1-z}\right) + \left(\frac{1-z}{1+z}\right) - 2}{\left(\frac{1+z}{1-z}\right) - \left(\frac{1-z}{1+z}\right)} \\
&= \left(\frac{2+2z^2}{1-z^2} - 2\right)\left(\frac{1-z^2}{4z}\right) \\
&= z
\end{aligned}$$

Arfken 6.1.21

(a) In general we have

$$\begin{aligned}
e^{\ln z} &= e^{\ln r + i\theta + i2\pi n} \\
&= r \cdot e^{i\theta} \cdot e^{i2\pi n} \\
&= r\, i\theta \\
&= z
\end{aligned}$$

(b) To do this problem it is important to realize that we can write a complex function, $f(z)$, in polar-like coordinates exactly as we do the complex variable $z = re^{i\theta}$. For example, we can write $f(z) = R(x,y)e^{i\Theta(x,y)}$.

In particular

$$\begin{aligned}
\ln e^z &= \ln f(z) \\
&= \ln\left(R(x,y)e^{i\Theta(x,y)}\right) \\
&= \ln R + \ln e^{i\Theta + i2\pi n} \\
&= \ln R + i\Theta + i2\pi n \\
&= z + i2\pi n \\
&\neq z
\end{aligned}$$

Arfken 6.2.3

If $w(z) = u(x,y) + iv(x,y)$ is analytic, then u and v each satisfy Laplace's equation. (This is done in 6.2.1. Take partial derivatives of the Cauchy Riemann conditions and equate.) This means

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

Recall that for a function to have a maximum or a minimum, the second derivatives of that function must be of the same sign (positive for a minimum and negative for a maximum). Because the functions u and v satisfy the Laplace equation, they cannot satisfy this condition for any point within the region in which $w(z)$ is analytic. Therefore, u and v (and hence $w(z)$) cannot have a maximum or a minimum within the region of analyticity.

Arfken 6.2.5

(a) For $u(x,y) = x^3 - 3xy^2$, the Cauchy Riemann conditions imply

$$\begin{aligned}
v(x,y) &= \int \frac{\partial u}{\partial x}\, dy \\
&= \int \left(3x^2 - 3y^2\right)\, dy \\
&= 3x^2 y - y^3 + a_0(x)
\end{aligned}$$

and

$$\begin{aligned}
v(x,y) &= -\int \frac{\partial u}{\partial y}\, dx \\
&= \int (6xy)\, dx \\
&= 3x^2 y + a_1(y)
\end{aligned}$$

For consistency, this leads to $a_0(x) = 0$ and $a_1(y) = -y^3$ and $w(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$.

(b) For $v(x,y) = e^{-y}\sin x$, the Cauchy Riemann conditions imply

$$\begin{aligned}
u(x,y) &= \int \frac{\partial v}{\partial y}\, dx \\
&= \int -e^{-y}\sin x\, dx \\
&= e^{-y}\cos x + a_0(y)
\end{aligned}$$

and

$$\begin{aligned}
u(x,y) &= -\int \frac{\partial v}{\partial x}\, dy \\
&= \int -e^{-y}\cos x\, dy \\
&= e^{-y}\cos x + a_1(y)
\end{aligned}$$

For consistency, this leads to $a_0(x) = 0$ and $a_1(y) = 0$ and $w(z) = e^{-y}\cos x + ie^{-y}\sin x$.

Arfken 6.2.7

The function $f(z)$ is analytic. The function $f(z^*)$ can be thought of as $f(z)$ with y replaced with $-y$. Likewise $f^*(z^*)$ can be thought of as $f(z^*)$ with i replace with $-i$. So, we have

$$\begin{aligned} f^*(z^*) &= u(x,-y) - iv(x,-y) \\ &= \hat{u}(x,y) + i\hat{v}(x,y) \end{aligned}$$

If $f^*(z^*)$ is to be analytic, the Cauchy Riemann conditions must be satisfied for it

$$\begin{aligned} \frac{\partial \hat{u}}{\partial x} &= \frac{\partial \hat{v}}{\partial y} \\ \frac{\partial u}{\partial x} &= (-1)\frac{\partial(-v)}{\partial y} \\ \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \hat{u}}{\partial y} &= -\frac{\partial \hat{v}}{\partial x} \\ (-1)\frac{\partial u}{\partial y} &= -\frac{\partial(-v)}{\partial x} \\ -\frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x} \end{aligned}$$

which are the Cauchy Riemann conditions for $f(z)$ which we know are satisfied since it is analytic. Therefore, $f^*(z^*)$ is analytic.

Arfken 6.3.2

Recall that the definition of a contour integral is

$$\int_c f(z)dz = \lim_{n\to\infty} \sum_{j=1}^{\infty} f(\zeta_j)(z_j - z_{j-1})$$

where ζ_j is a point on the contour between z_j and z_{j-1} and the contour is assumed specified. We can now construct the following inequality if we replace the value of the complex function $f(\zeta_j)$ with the maximum value of $f(z)$ along the curve

$$\begin{aligned} \left| \int_c f(z)dz \right| &< \lim_{n\to\infty} \sum_{j=1}^{\infty} |f|_{\max}(z_j - z_{j-1}) \\ &= |f|_{\max} \lim_{n\to\infty} \sum_{j=1}^{\infty} (z_j - z_{j-1}) \\ &= |f|_{\max} \lim_{n\to\infty} L \\ &= |f|_{\max} L \end{aligned}$$

where L is the length of the contour, C.

Arfken 6.3.4

The integral $\oint_c dz/(z^2+z)$ cannot be evaluated for any contour enclosing the origin by using the Cauchy integral theorem because the function being integrated is not analytic at $z=0$ (or at $z=-1$). However, it is zero if we consider a circle for C with $|z|=R>1$:

$$\begin{aligned}
\oint \frac{dz}{z^2+z} &= \int_0^{2\pi} \frac{Re^{i\theta} i d\theta}{R^2 e^{i2\theta} + Re^{i\theta}} \\
&= \int_0^{2\pi} \frac{iR^{-i\theta}+i}{(Re^{i\theta}+1)(Re^{-i\theta}+1)}\, d\theta \\
&= \int_0^{2\pi} \frac{R\sin\theta + i(1+R\cos\theta)}{R^2+2R\cos\theta+1}\, d\theta
\end{aligned}$$

The real part of the integral can be shown to be zero (substitute $\xi = \theta - \pi$ and show that it is an odd integral over a symmetric interval – which must be zero) while the imaginary part can be done most easily by looking it up in a table, or better yet by using Mathematica or Maple. It is then straightforward to show the integral must also be zero.

You have to be careful with this problem or you can come away with the wrong impression. It is important to realize that this particular integral is not zero *because* of the Cauchy integral theorem (after all, it violates the assumptions of that theorem for any contour, C, that encloses any nonanlytic point of the integrand, in this case $z=0,-1$) but *in spite* of it.