Assignment 7

Arfken 6.4.1

Taking the contour C to be a circle of radius r centered at z_0, we can use $re^{i\theta} = z - z_0$ to calculuate

$$\begin{aligned}
\oint (z-z_0)^n \, dz &= \int_0^{2\pi} r^n e^{in\theta} \, ire^{i\theta} d\theta \\
&= i \begin{cases} \frac{r^{n+1}}{n+1} \, e^{i(n+1)\theta} \Big|_0^{2\pi} & n \neq -1 \\ \theta \Big|_0^{2\pi} & n = -1 \end{cases} \\
&= \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}
\end{aligned}$$

Note that though we have taken C to be a circle, because $(z-z_0)^n$ is analytic at all other points in the plane, C could be deformed to be anything in the complex plane.

Arfken 6.4.2

Evaluate the contour integral with a circular contour, C of radius 1. By the argument from the previous problem, we can always deform this contour to any in the plane encircling $z = 0$ (the single point of non-analyticity). We have

$$\begin{aligned}
\frac{1}{2\pi i} \oint z^{m-n-1} dz &= \frac{1}{2\pi i} \int_0^{2\pi} r^{m-n-1} e^{i\theta(m-n-1)} re^{i\theta} id\theta \\
&= \frac{1}{2\pi} \begin{cases} \frac{r^{m-n}}{i(m-n)} e^{i\theta(m-n)} \Big|_0^{2\pi} & n \neq m \\ \theta \Big|_0^{2\pi} & n = m \end{cases} \\
&= \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases} \\
&= \delta_{mn}
\end{aligned}$$

which is the Kronecker delta.

Arfken 6.4.6

The function $f(z)$ is analytic on and in C. From the Cauchy integral formula, we have

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z'-z} \, dz'$$

Taking the derivative of this with respect to z, we get

$$f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z)^2} \, dz'$$

We will assume that a similar result holds for the n^{th} derivative:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(z')}{(z'-z)^{n+1}} \, dz'$$

Taking another derivative with respect to z (*not* z'!), we can establish this for $n+1$

$$f^{(n+1)}(z) = \frac{(n+1)!}{2\pi i} \oint_C \frac{f(z')}{(z'-z)^{n+2}}\, dz'$$

Therefore since it is true for $n=1$, assumed for n and shown for $n+1$, by induction, this is true.

Arfken 6.4.8

We can use the Cauchy integral formula for the n^{th} derivative to convert Rodrigues formulae into Schlaefli integrals.

For the Legendre polynomials, we have

$$\begin{aligned}
P_n(x) &= \frac{1}{2^n\, n!} \frac{d^n}{dx^n}(x^2-1)^n \\
&= \frac{1}{2^n\, n!} f^{(n)}(x) \\
&= \frac{1}{2^n\, n!} \frac{n!}{2\pi i} \oint_C \frac{f(z')}{(z'-x)^{n+1}}\, dz' \\
&= \frac{1}{2^n\, 2\pi i} \oint_C \frac{(z'^2-1)^n}{(z'-x)^{n+1}}\, dz' \\
&= \frac{(-1)^n}{2^n} \cdot \frac{1}{2\pi i} \oint_C \frac{(1-z'^2)^n}{(z'-x)^{n+1}}\, dz'
\end{aligned}$$

For the Hermite polynomials, we get

$$\begin{aligned}
H_n(x) &= (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \\
&= (-1)^n e^{x^2} f^{(n)}(x) \\
&= (-1)^n e^{x^2} \frac{n!}{2\pi i} \oint_C \frac{f(z')}{(z'-x)^{n+1}}\, dz' \\
&= (-1)^n e^{x^2} \frac{n!}{2\pi i} \oint_C \frac{e^{-z'^2}}{(z'-x)^{n+1}}\, dz' \\
&= (-1)^n e^{x^2} \frac{n!}{2\pi i} \oint_C \frac{e^{-(-t+x)^2}}{(-t)^{n+1}} \cdot (-dt) \qquad \text{where} \quad t = x - z' \\
&= \frac{n!}{2\pi i} \oint_C t^{-n-1} e^{-t^2+2tx}\, dt
\end{aligned}$$

For the Laguerre polynomials, we get

$$\begin{aligned}
L_n(x) &= \frac{e^x}{n!} \frac{d^n}{dx^n}\left(x^n e^{-x}\right) \\
&= \frac{e^x}{n!} f^{(n)}(x) \\
&= \frac{e^x}{n!} \frac{n!}{2\pi i} \oint_C \frac{f(z')}{(z'-x)^{n+1}}\, dz' \\
&= \frac{e^x}{2\pi i} \oint_C \frac{z'^n e^{-z'}}{(z'-x)^{n+1}}\, dz'
\end{aligned}$$

which, if we use $t = 1 - x/z'$, we can convert into a slightly more standard form

$$L_n(x) = \frac{1}{2\pi i} \oint_C \frac{e^{-xt/(1-t)}}{(1-t)\, t^{n+1}}\, dt$$

Arfken 6.5.3

Our assumptions are that $f(z)$ is analytic and $|f(z)| \leq 1$ for $|z| \leq 1$ and that $f(0) = 0$. First, we have from the Cauchy integral formula for C the unit circle

$$\oint_C \frac{f(z)}{z}\,dz = 2\pi i f(0) = 0$$

Because this will be true for all contours, C, inside the unit circle and encircling the origin $z = 0$, we conclude from Morera's theorem that $f(z)/z$ is analytic on $|z| \leq 1$. Using the Cauchy integral formula again, we have

$$\left[\frac{f(z)}{z}\right]^n = \frac{1}{2\pi i}\oint_C \left[\frac{f(z')}{z'}\right]^n \frac{1}{z'-z}\,dz'$$

where we again take C to be the unit circle. Taking magnitudes, we can finish our proof

$$\begin{aligned}
\left|\left[\frac{f(z)}{z}\right]^n\right| &= \left|\frac{1}{2\pi i}\oint_C \left[\frac{f(z')}{z'}\right]^n \frac{1}{z'-z}\,dz'\right| \\
&\leq \frac{1}{2\pi}\left|\frac{1}{z'^n}\right|_{\max}\left|\oint_C \frac{f(z')^n}{z'-z}\,dz'\right| \\
&\leq \frac{1}{2\pi}\left|\frac{1}{z'^n}\right|_{\max} \cdot \left|2\pi i f(z)^n\right|_{\max} \\
&\leq \left(|f(z)|_{\max}\right)^n \\
&\leq 1
\end{aligned}$$

where in the second line we have used a variant of the Darboux inequality (problem 6.3.2) and in the third line we have used the fact that the power of an analytic function will also be analytic. Taking the n^{th} root and rearranging, we get our answer

$$|f(z)| \leq |z|$$

Arfken 6.5.7

The function $f(z)$ is analytic in a region that includes the real axis and is purely imaginary if z is real. Because $f(z)$ is analytic, we can express it in terms of a Taylor series around some point on the real axis, x_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - x_0)^n$$

Being purely imaginary on the x-axis yields

$$f(x) = -f^*(x) = -\sum_{n=0}^{\infty} a_n^* (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

which implies that the coefficients a_n are purely imaginary. For a general point off the axis, we have

$$\begin{aligned}
f(z^*) &= \sum_{n=0}^{\infty} a_n (z^* - x_0)^n \\
&= \sum_{n=0}^{\infty} -a_n^* \left((z - x_0)^n\right)^* \\
&= -\left(\sum_{n=0}^{\infty} a_n (z - x_0)^n\right)^* \\
&= -\left[f(z)\right]^*
\end{aligned}$$

(b) For $f(z) = iz = ix - y$, it is straightforward to see that $f(z^*) = iz^* = ix + y$ and $f^*(z) = -iz^* = -ix - y$.

Arfken 6.5.8

The Laurent series for $f(z) = (e^z - 1)^{-1}$ about the origin is given by

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \\ a_n &= \frac{1}{2\pi i} \oint_C \frac{f(z')\, dz'}{(z' - z_0)^{n+1}} \end{aligned}$$

Calculating the coefficients, a_n using the unit circle as C with $z_0 = 0$, we have

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_C \frac{dz'}{z'^{n+1}(e^{z'} - 1)} \\ &= \frac{1}{2\pi i} \oint_C \frac{dz'}{z'^{n+1}} \sum_{k=0}^{\infty} \frac{B_k z'^{k-1}}{k!}\, dz' \\ &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \frac{B_k}{k!} \oint_C z'^{k-n-1-1}\, dz' \\ &= \sum_{k=0}^{\infty} \frac{B_k}{k!} \delta_{k,n+1} \\ &= \begin{cases} \frac{B_{n+1}}{(n+1)!} & n \geq -1 \\ 0 & n < -1 \end{cases} \end{aligned}$$

where we have used the series definition of the Bernoulli numbers and the result of problem 6.4.2 to define the Kronecker delta in the last line.

The Laurent expansion is now

$$\begin{aligned} \frac{1}{e^z - 1} &= \sum_{n=-1}^{\infty} \frac{B_{n+1}}{(n+1)!} z^n \\ &= \frac{B_0}{z} + B_1 + \frac{B_2}{2} z + \cdots \\ &= \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + \cdots \end{aligned}$$

taking the first three terms.

Arfken 6.5.9

We want to prove that a Laurent expansion about a point z_0 is unique. We can do this most easily by induction. Assuming that

$$f(z) = \sum_{n=-N}^{\infty} a_n (z - z_0)^n = \sum_{n=-N}^{\infty} b_n (z - z_0)^n$$

multiply both sides by $(z - z_0)^N$ and set $z = z_0$. The result is

$$a_{-N} = b_{-N}$$

Continue this process by multiplying the sums by $(z - z_0)^N$, taking a derivative and setting $z = z_0$. This results in

$$a_{-N+1} = b_{-N+1}$$

Assume now that $a_k = b_k$ for all $k \geq -N$ with $k > 0$, we will show it for $a_{k+1} = b_{k+1}$. Because of this assumption, we can cancel the first terms up to $k+1$. The series then read

$$\sum_{n=k+1}^{\infty} a_n (z - z_0)^n = \sum_{n=k+1}^{\infty} b_n (z - z_0)^n$$

Taking $k+1$ derivatives and setting $z = z_0$, we get

$$(k+1)!\, a_{k+1} = (k+1)!\, b_{k+1}$$

and our proof via induction is done and the Laurent expansions are unique.

Arfken 6.5.10

For the function $f(z) = [z(z-1)]^{-1}$ the Laurnet expansion about $z = 1$ for small values of $|z-1|$ is given by $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-1)^n$ where a_n is found via a contour integral using a circle of radius r centered at $z = 1$ (*i.e.* $z - 1 = re^{i\theta}$) as our contour C.

$$\begin{aligned}
a_n &= \frac{1}{2\pi i} \oint_C \frac{f(z')\, dz'}{(z' - z_0)^{n+1}} \\
&= \frac{1}{2\pi i} \oint_C \frac{dz'}{(z'-1)^{n+2}\, z'} \\
&= \frac{1}{2\pi i} \oint_C \frac{1}{(z'-1)^{n+2}} \frac{1}{1-(1-z')}\, dz' \\
&= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \oint_C \frac{(1-z')^k}{(z'-1)^{n+2}} \\
&= \frac{1}{2\pi i} \sum_{k=0}^{\infty} (-1)^k r^{k-n-2} \int_0^{2\pi} e^{i(k-n-2)\theta} r i e^{i\theta} d\theta \\
&= \sum_{k=0}^{\infty} (-1)^k r^{k-n-1}\, \delta_{k,n+1} \\
&= \begin{cases} (-1)^{n+1} & n \geq -1 \\ 0 & n < -1 \end{cases}
\end{aligned}$$

Thus the Laurent expansion is

$$\frac{1}{z\,(z-1)} = -\sum_{n=-1}^{\infty} (1-z)^n$$

This expansion will hold up to the next singularity at $z = 0$. Hence, the radius of convergence of this Laurent expansion is $|z| < 1$.

(b) The Laurent expansion for this function around $z = 1$ but for $|z - 1|$ very large can be found in a similar way,

$$
\begin{aligned}
a_n &= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)^{n+1}} \\
&= \frac{1}{2\pi i}\oint_C \frac{dz'}{(z'-1)^{n+2}\,z'} \\
&= \frac{1}{2\pi i}\oint_C \frac{1}{(z'-1)^{n+2}}\,\frac{1}{(z'-1)+1}\,dz' \\
&= \frac{1}{2\pi i}\oint_C \frac{1}{(z'-1)^{n+3}}\,\frac{1}{1+(z'-1)^{-1}}\,dz' \\
&= \frac{1}{2\pi i}\sum_{k=0}^{\infty}\oint_C \frac{(1-z')^{-k}}{(z'-1)^{n+3}}\,dz' \\
&= \frac{1}{2\pi i}\sum_{k=0}^{\infty}(-1)^k r^{-k-n-3}\int_0^{2\pi} e^{i(-k-n-3)\theta} r i e^{i\theta} d\theta \\
&= \sum_{k=0}^{\infty}(-1)^k r^{-k-n-2}\,\delta_{-k,n+2} \\
&= \begin{cases} (-1)^n & n \le -2 \\ 0 & n > -2 \end{cases}
\end{aligned}
$$

Thus the Laurent expansion is

$$
\begin{aligned}
\frac{1}{z\,(z-1)} &= \sum_{n=-\infty}^{-2} (1-z)^n \\
&= \sum_{n=2}^{\infty} \frac{1}{(1-z)^n}
\end{aligned}
$$

Arfken 6.6.1 A function has a pole of order m at $z = z_0$. Find the coefficient of $(z-z_0)^{-1}$, a_{-1}.

The Laurent expansion for $f(z)$ near z_0 will be given by

$$
\begin{aligned}
f(z) &= \sum_{n=-m}^{\infty} a_n (z-z_0)^n \\
&= \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + \cdots
\end{aligned}
$$

To read off any given constant coefficient we need to eliminate all of the z dependance. For instance to get a_{-m}, multiply by $(z-z_0)^m$ and set everything to $z = z_0$:

$$
(z-z_0)f(z)\Big|_{z=z_0} = a_{-m}
$$

To get higher order coefficients, in addition to the multiplication, we need to take derivatives, *e.g.*

$$
a_{-m+1} = \frac{d}{dz}\big((z-z_0)f(z)\big)\Big|_{z=z_0}
$$

Finally, to get the residue, we take $m-1$ derivatives

$$
a_{-1} = \frac{1}{(n-1)!}\frac{d^{m-1}}{dz^{m-1}}\big((z-z_0)f(z)\big)\Big|_{z=z_0}
$$

Arfken 6.6.2 A function $f(z)$ is a quotient of two everywhere analytic functions, $f_1(z)$ and $f_2(z)$. Further there is a pole in $f(z)$ where $f_2(z_0) = 0$. $f_1(z_0) \neq 0$ and $f_2'(z_0) \neq 0$. Find the residue.

Since $f_2(z_0) = 0$, we can write $f_2(z) = (z - z_0)g(z)$ where $g(z)$ is some function without a zero at z_0. To verify this, take the derivative: $f_2'(z) = g(z) + (z - z_0)g'(z)$. At $z = z_0$, this becomes $f_2'(z_0) = g(z_0)$which by assumption is not zero. Thus $f_2(z)$ has a single (simple) zero at z_0 and thus $f(z)$ has a simple pole at z_0. We can then write $f(z)$ as

$$f(z) = \frac{f_1(z)}{(z - z_0)g(z)}$$

The residue of this function is just $f(z)(z - z_0)$ evaluated at $z = z_0$. Thus we get

$$\begin{aligned} a_{-1} &= \frac{f_1(z_0)}{g(z_0)} \\ &= \frac{f_1(z_0)}{f_2'(z_0)} \end{aligned}$$