Assignment 8

(1) Evaluate $I = \int_0^{2\pi}(a^2+\sin^2\theta)^{-2}d\theta$ using contour integrals. Because this is a function of trigonometric functions, we can take a contour, C, around the unit circle and replace $\sin\theta$ with $(z-1/z)/(2i)$. Thus, I is exactly this contour integral around the unit circle

$$\begin{aligned} I = \oint_C \frac{1}{(a^2-(z-1/z)^2/4)^2}\frac{dz}{iz} &= \oint_C \frac{2^4z^4}{\left(4a^2z^2-(z^2-1)^2\right)^2}\frac{dz}{iz} \\ &= \frac{16}{i}\oint_C \frac{z^3\,dz}{\left((z^2-1)-2az\right)^2\left((z^2-1)+2az\right)^2} \\ &= \frac{16}{i}\oint_C \frac{z^3\,dz}{(z-z_1)^2(z-z_2)^2(z-z_3)^2(z-z_4)^2} \end{aligned}$$

where $z_{1,2} = a\pm\sqrt{a^2+1}$ and $z_{3,4} = -a\pm\sqrt{a^2+1}$ are all second order poles of our function. Because $a>1$, the poles $z_1 = a+\sqrt{a^2+1}$ and $z_4 = -a-\sqrt{a^2+1}$ lie *outside* the unit circle and should not be included in the calculation of the residues. Also, note that $z_4 = -z_1$ and $z_3 = -z_2$

From the residue theorem, this contour integral becomes

$$\begin{aligned} I &= \frac{16}{i}2\pi i\left\{\frac{d}{dz}\left(\frac{z^3}{(z-z_1)^2(z-z_3)^2(z-z_4)^2}\right)\bigg|_{z=z_2} + \frac{d}{dz}\left(\frac{z^3}{(z-z_1)^2(z-z_2)^2(z-z_4)^2}\right)\bigg|_{z=z_3}\right\} \\ &= 32\pi\{\quad 3{z_2}^2(z_2-z_1)^{-2}(z_2-z_3)^{-2}(z_2-z_4)^{-2} - 2{z_2}^3(z_2-z_1)^{-3}(z_2-z_3)^{-2}(z_2-z_4)^{-2} \\ &\quad - 2{z_2}^3(z_2-z_1)^{-2}(z_2-z_3)^{-3}(z_2-z_4)^{-2} - 2{z_2}^3(z_2-z_1)^{-2}(z_2-z_3)^{-2}(z_2-z_4)^{-3} \\ &\quad + 3{z_3}^2(z_3-z_1)^{-2}(z_3-z_2)^{-2}(z_3-z_4)^{-2} - 2{z_3}^3(z_3-z_1)^{-3}(z_3-z_2)^{-2}(z_3-z_4)^{-2} \\ &\quad - 2{z_3}^3(z_3-z_1)^{-2}(z_3-z_2)^{-3}(z_3-z_4)^{-2} - 2{z_3}^3(z_3-z_1)^{-2}(z_3-z_2)^{-2}(z_3-z_4)^{-3}\quad\} \\ &= 32\pi{z_2}^2(z_2-z_1)^{-2}(2z_2)^{-2}(z_2+z_1)^{-2} \\ &\quad \{\quad 3-2z_2(z_2-z_1)^{-1}-2z_2(2z_2)^{-1}-2z_2(z_2+z_1)^{-1} \\ &\quad +3+2z_2(-z_2-z_1)^{-1}+2z_2(-2z_2)^{-1}+2z_2(-z_2+z_1)^{-1}\} \\ &= \frac{32\pi}{({z_2}^2-{z_1}^2)^2}\left[1-\frac{z_2(z_2+z_1)+z_2(z_2-z_1)}{({z_2}^2-{z_1}^2)}\right] \\ &= \frac{32\pi}{({z_2}^2-{z_1}^2)^3}\left[-{z_2}^2-{z_1}^2\right] \end{aligned}$$

Using ${z_1}^2+{z_2}^2 = 4a^2+2$ and ${z_1}^2-{z_2}^2 = 4a\sqrt{a^2+1}$, this becomes

$$I = \int_0^{2\pi}\frac{d\theta}{(a^2+\sin^2\theta)^2} = \frac{\pi}{a^3}\,\frac{2a^2+1}{(a^2+1)^{3/2}}$$

(2) Evaluate $I = \int_0^\infty x/(1+x^3)\,dx$. We want a contour that includes the positive half of the real axis. The denominator has zeros at $-1 = e^{i\pi}, e^{i\pi/3}$, and $e^{-i\pi/3}$. Note that including the negative real axis will give us a contribution to the contour integral quite different from I, so it would be wise to avoid using that as part of the contour. We might also imagine that part of our contour will be at least some part of a circle with fixed radius, R, that will eventually go to ∞. The best choice is hinted at by the x^3 in the denominator. Take as our contour the following three parts: (1) the positive real axis, (2) the large sector from $\theta=0$ to $\theta=2\pi/3$ and (3) the line or ray with constant $\theta = 2\pi/3$. This contour encloses a third of the complex plane (in the limit that the radius, R, in part (2) goes to ∞) as well as the pole at $z=e^{i\pi/3}$. The reason we take part (3) is that on the return trip along the ray, the denominator in the complex plane, $1+z^3 = 1+(re^{i2\pi/3})^3 = 1+r^3$

and we will recover a real integral similar to I. We now have the contour integral

$$\oint_C \frac{z}{1+z^3}\,dz = \lim_{R\to\infty}\left\{\int_0^R \frac{r}{1+r^3}\,dr + \int_0^{2\pi/3} \frac{Re^{i\theta}}{1+R^3e^{3i\theta}}\,Rie^{i\theta}d\theta + \int_R^0 \frac{re^{i2\pi/3}}{1+(re^{i2\pi/3})^3}\,e^{i2\pi/3}dr\right\}$$
$$= \int_0^\infty \frac{r}{1+r^3}\,dr - e^{i4\pi/3}\int_0^\infty \frac{r}{1+r^3}\,dr$$
$$= -2ie^{i2\pi/3}\sin(2\pi/3)\,I$$

where in the second line, the second term goes to zero.

Now using the residue theorem, we evaluate the contour integral

$$\oint_C \frac{z}{1+z^3}\,dz = 2\pi i\,\mathrm{Res}\left(\frac{z}{1+z^3}\right)\Bigg|_{e^{i\pi/3}}$$
$$= 2\pi i\left(\frac{z}{3z^2}\right)\Bigg|_{e^{i\pi/3}}$$
$$= \frac{2\pi i}{3\,e^{i\pi/3}}$$

Combining these results, we get

$$I = \int_0^\infty \frac{x}{(1+x^3)}dx = \frac{\pi}{3\sin(2\pi/3)} = \frac{2\sqrt{3}}{9}\,\pi$$

(3) This integral, $I = \int_0^\infty \sinh(ax)/\sinh(\pi x)\,dx$, can be written with limits from $-\infty$ to ∞ provided we multiply the result by $1/2$. We can do this since the integrand is an even function. The poles of the function sit along the imaginary axis at all integer multiples of i with the exception of $z=0$. One possibility for a good contour is a rectangle that extends from $(-R,0)$ to $(R,0)$ along the real axis, from $(R,0)$ to $(R,1)$ along the line $x=R$, from $(R,1)$ to $(-R,1)$ along the line $y=1$, and then from $(-R,1)$ to $(-R,0)$ along the line $x=-R$. Taking this as our contour, we get

$$\oint_C \frac{\sinh(az)}{\sinh(\pi z)}\,dz = \lim_{R\to\infty}\left\{\int_{-R}^R \frac{\sinh(ax)}{\sinh(\pi x)}\,dx + \int_0^1 \frac{\sinh\big(a(R+iy)\big)}{\sinh\big(\pi(R+iy)\big)}\,dy\right.$$
$$\left. + \int_R^{-R} \frac{\sinh\big(a(x+i)\big)}{\sinh\big(\pi(x+i)\big)}\,dx + \int_1^0 \frac{\sinh\big(a(-R+iy)\big)}{\sinh\big(\pi(-R+iy)\big)}\,dy\right\}$$

The integrals over y which are the vertical end pieces of the contour, C, with R fixed go to 0 as $R\to\infty$ provided $a<\pi$. This is seen by considering only the integrands

$$\lim_{R\to\infty}\frac{\sinh\big(a(\pm R+iy)\big)}{\sinh\big(\pi(\pm R+iy)\big)} = \lim_{R\to\infty}\frac{e^{a(\pm R+iy)} - e^{-a(\pm R+iy)}}{e^{\pi(\pm R+iy)} - e^{-\pi(\pm R+iy)}}$$
$$= \lim_{R\to\infty} e^{R(a-\pi)}$$
$$= 0 \quad \text{if} \quad a<\pi$$

Our contour integral now is

$$\oint_C \frac{\sinh(az)}{\sinh(\pi z)}\,dz = \int_{-\infty}^\infty \frac{\sinh(ax)}{\sinh(\pi x)}\,dx + \int_\infty^{-\infty} \frac{\sinh\big(a(x+i)\big)}{\sinh\big(\pi(x+i)\big)}\,dx$$
$$= \int_{-\infty}^\infty \frac{\sinh(ax)}{\sinh(\pi x)}\,dx + \int_\infty^{-\infty} \frac{\sinh(ax)\cosh(ia)+\cosh(ax)\sinh(ia)}{\sinh(\pi x)\cosh(i\pi)+\cosh(\pi x)\sinh(i\pi)}\,dx$$
$$= \int_{-\infty}^\infty \frac{\sinh(ax)}{\sinh(\pi x)}\,dx + \cos(a)\int_{-\infty}^\infty \frac{\sinh(ax)}{\sinh(\pi x)}\,dx + i\sin(a)\int_{-\infty}^\infty \frac{\cosh(ax)}{\sinh(\pi x)}\,dx$$

where we have used $\cosh(i\pi) = -1$ and $\sinh(i\pi) = 0$. Now note that the third integral is zero because we are integrating an odd function over a symmetric interval. The second integral is just some constant times I. Now, using the residue theorem, we can evaluate the contour integral. The only pole that we need consider is that at $z = i$. In fact, it sits on the contour; and as a simple pole, we need take only half of its contribution. The point $z = 0$, it should be mentioned, is not a pole. This can be seen by taking the limit of the function as $z \to 0$. The result is a finite value, a/π. The contour integral is then

$$\begin{aligned}\oint_C \frac{\sinh(az)}{\sinh(\pi z)}\,dz &= \pi i \operatorname{Res}\left(\frac{\sinh(az)}{\sinh(\pi z)}\right)\bigg|_{z=i} \\ &= i\pi\,\frac{\sinh(ia)}{\pi\cosh(i\pi)} \\ &= \sin(a)\end{aligned}$$

Putting it all together, we have

$$I = \int_0^\infty \frac{\sinh(ax)}{\sinh(\pi x)}\,dx = \frac{1}{2}\,\frac{\sin a}{1+\cos a}$$

(4) The integral, $I = \int_{-\infty}^{\infty} e^{ax}/\cosh(x)\,dx$, with $0 < a < 1$ is very similar to the previous problem since both deal with exponentials. Since the poles of the function lie at half integer multiples of π on the imaginary axis: $z = i(2n+1)\pi/2$, we can use a contour similar to that in problem 3. The only difference will be to take the top of the contour to run from (R, π) to $(-R, \pi)$ along the line $y = \pi$ instead of using $y = 1$. In this case only a single pole is entirely enclosed in the contour. Taking this as our contour, we get

$$\begin{aligned}\oint_C \frac{e^{az}}{\cosh z}\,dz = \lim_{R\to\infty}\Bigg\{&\int_{-R}^{R}\frac{e^{ax}}{\cosh x}\,dx + \int_0^{\pi/2}\frac{e^{a(R+iy)}}{\cosh\big(R+iy\big)}\,dy \\ &+ \int_R^{-R}\frac{e^{a(x+i\pi)}}{\cosh\big(x+i\pi\big)}\,dx + \int_1^0 \frac{e^{a(-R+iy)}}{\cosh\big(-R+iy\big)}\,dy\Bigg\}\end{aligned}$$

As in the previous problem, the integrals over y which are the vertical end pieces of the contour, C, with R fixed go to 0 as $R \to \infty$ provided $a < 1$. The proof of this is exactly as before.

The contour integral is now

$$\begin{aligned}\oint_C \frac{e^{az}}{\cosh z}\,dz &= \int_{-\infty}^{\infty}\frac{e^{ax}}{\cosh x}\,dx + \int_{\infty}^{-\infty}\frac{e^{a(x+i\pi)}}{\cosh\big(x+i\pi\big)}\,dx \\ &= \int_{-\infty}^{\infty}\frac{e^{ax}}{\cosh x}\,dx + e^{ia\pi}\int_{\infty}^{-\infty}\frac{e^{ax}}{\cosh x\cosh(i\pi) - \sinh x\sinh(i\pi)}\,dx \\ &= \int_{-\infty}^{\infty}\frac{e^{ax}}{\cosh x}\,dx + e^{ia\pi}\int_{-\infty}^{\infty}\frac{e^{ax}}{\cosh x}\,dx\end{aligned}$$

where we have used $\cosh(i\pi) = -1$ and $\sinh(i\pi) = 0$. The second integral is just some constant times I. Now, using the residue theorem, we can evaluate the contour integral. The only pole that we need consider is that at $z = i\pi/2$. The contour integral is then

$$\begin{aligned}\oint_C \frac{e^{az}}{\cosh z}\,dz &= 2\pi i \operatorname{Res}\left(\frac{e^{az}}{\cosh z}\right)\bigg|_{z=i} \\ &= 2i\pi\,\frac{e^{ia\pi/2}}{\sinh(i\pi/2)} \\ &= 2\pi\,e^{ia\pi/2}\end{aligned}$$

Putting it all together, we have

$$I = \int_{-\infty}^{\infty}\frac{e^{ax}}{\cosh x}\,dx = \frac{\pi}{\cos\big(\pi a/2\big)}$$

(5) To find the sum, $S = \sum_{n=1}^{\infty}(n^2 - a^2)^{-1}$, use a contour, C, as from class, that avoids all the poles on the positive and negative real axis. The contour integral with this contour can be considered in four parts: (1) a semicircle in the upper half plane, (2) a contour (closed) around the poles at negative integers, (3) a semicircle in the lower half plane, and (4) a contour (closed) around the poles at the positive integers. Parts (1) and (3) will go to zero in the limit the radii of the semicircles go to ∞. Parts (2) and (4) give sums over all the integers because the function $\pi\cot(\pi z)$ is chosen to give poles with residue 1 at all integer values (in addition to the contribution from the poles at $z = \pm a$. So we get

$$\begin{aligned}\oint_C \frac{1}{z^2-a^2}\frac{\pi\cos(\pi z)}{\sin(\pi z)}\,dz &= -2\pi i\sum_{n=-\infty}^{-1}\frac{1}{n^2-a^2} - 2\pi i\,\mathrm{Res}\left(\frac{1}{z^2-a^2}\pi\cot(\pi z)\right)\Bigg|_{z=-a} \\ &\quad - 2\pi i\sum_{n=1}^{\infty}\frac{1}{n^2-a^2} - 2\pi i\,\mathrm{Res}\left(\frac{1}{z^2-a^2}\pi\cot(\pi z)\right)\Bigg|_{z=a} \\ &= -4\pi i\sum_{n=1}^{\infty}\frac{1}{n^2-a^2} - 2\pi i\,\frac{2\pi\cot(\pi a)}{2a}\end{aligned}$$

Evaluating the original contour integral with the residue theorem, the only pole we need consider is that at $z = 0$. No other poles are contained within the full contour, C. Thus we have

$$\begin{aligned}\oint_C \frac{1}{z^2-a^2}\frac{\pi\cos(\pi z)}{\sin(\pi z)}\,dz &= 2\pi i\,\mathrm{Res}\left(\frac{1}{z^2-a^2}\frac{\pi\cos(\pi z)}{\sin(\pi z)}\right)\Bigg|_{z=0} \\ &= -\frac{2\pi i}{a^2}\end{aligned}$$

Equating these two results we find

$$\sum_{n=1}^{\infty}\frac{1}{n^2-a^2} = \frac{1}{2a^2} - \frac{\pi}{2a}\cot(\pi a)$$

(6) To find the sum, $S = \sum_{n=1}^{\infty}(-1)^n(n^2 - a^2)^{-1}$, use the same contour from **(5)** but drop the $\cos(\pi z)$ in the contour integral since we want an alternating series. Otherwise, this is very similar to the previous problem. Applying it here, we get

$$\begin{aligned}\oint_C \frac{1}{z^2-a^2}\frac{\pi}{\sin(\pi z)}\,dz &= -2\pi i\sum_{n=-\infty}^{-1}\frac{(-1)^n}{n^2-a^2} - 2\pi i\,\mathrm{Res}\left(\frac{1}{z^2-a^2}\pi\csc(\pi z)\right)\Bigg|_{z=-a} \\ &\quad - 2\pi i\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-a^2} - 2\pi i\,\mathrm{Res}\left(\frac{1}{z^2-a^2}\pi\csc(\pi z)\right)\Bigg|_{z=a} \\ &= -4\pi i\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-a^2} - 2\pi i\,\frac{2\pi\csc(\pi a)}{2a}\end{aligned}$$

Evaluating the original contour integral with the residue theorem, the only pole we need consider is that at $z = 0$. No other poles are contained within the full contour, C. Thus we have

$$\begin{aligned}\oint_C \frac{1}{z^2-a^2}\frac{\pi}{\sin(\pi z)}\,dz &= 2\pi i\,\mathrm{Res}\left(\frac{1}{z^2-a^2}\frac{\pi}{\sin(\pi z)}\right)\Bigg|_{z=0} \\ &= -\frac{2\pi i}{a^2}\end{aligned}$$

Equating these two results we find

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-a^2} = \frac{1}{2a^2} - \frac{\pi}{2a\sin(\pi a)}$$

(7) The sum $S = \sum_{n=1}^{\infty} (-1)^n (n^2 - a^2)^{-p}$ will, of course, be similar to the previous problem. The twist here is that the poles at $z = \pm a$ will be poles of order p. But, otherwise, we can take over our approach from the previous problem. We have

$$\begin{aligned}
\oint_C \frac{1}{(z^2-a^2)^p}\frac{\pi}{\sin(\pi z)}\,dz &= -2\pi i \sum_{n=-\infty}^{-1} \frac{(-1)^n}{(n^2-a^2)^p} - 2\pi i \,\mathrm{Res}\left(\frac{1}{(z^2-a^2)^p}\frac{\pi}{\sin(\pi z)}\right)\Bigg|_{z=-a} \\
&\quad - 2\pi i \sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2-a^2)^p} - 2\pi i \,\mathrm{Res}\left(\frac{1}{(z^2-a^2)^p}\frac{\pi}{\sin(\pi z)}\right)\Bigg|_{z=a} \\
&= -4\pi i \sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2-a^2)^p} - 2\pi i \frac{\pi}{2a\sin(\pi a)} \\
&= -4\pi i \sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2-a^2)^p} \\
&\quad - \frac{2\pi^2 i}{(p-1)!}\left[\frac{d^{p-1}}{dz^{p-1}}\left(\frac{\csc(\pi z)}{(z-a)^p}\right)\Bigg|_{z=-a} + \frac{d^{p-1}}{dz^{p-1}}\left(\frac{\csc(\pi z)}{(z+a)^p}\right)\Bigg|_{z=a}\right]
\end{aligned}$$

Evaluating the original contour integral with the residue theorem, the only pole we need consider is that at $z = 0$. No other poles are contained within the full contour, C. Thus we have

$$\begin{aligned}
\oint_C \frac{1}{(z^2-a^2)^p}\frac{\pi}{\sin(\pi z)}\,dz &= 2\pi i \,\mathrm{Res}\left(\frac{1}{(z^2-a^2)^p}\frac{\pi}{\sin(\pi z)}\right)\Bigg|_{z=0} \\
&= \frac{2\pi i}{(-a^2)^p}
\end{aligned}$$

Equating these two results we find

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2-a^2)^p} = \frac{(-1)^{p+1}}{2a^{2p}} - \frac{\pi}{2(p-1)!}\left[\frac{d^{p-1}}{dz^{p-1}}\left(\frac{\csc(\pi z)}{(z-a)^p}\right)\Bigg|_{z=-a} + \frac{d^{p-1}}{dz^{p-1}}\left(\frac{\csc(\pi z)}{(z+a)^p}\right)\Bigg|_{z=a}\right]$$

This is a general formula which we could work out for general p. But to save on the headache, let's take $p = 2$. Working this out, we find

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2-a^2)^2} = -\frac{1}{2a^4} + \frac{\pi}{4a^3}\frac{1}{\sin(\pi a)}\big(1 + \pi a \cot(\pi a)\big)$$

(8) For the sum, $S = \sum_{n=1}^{\infty} (n^2 - a^2)^{-1}(n^2 - b^2)^{-1}$, use the usual contour for the integral

$$\begin{aligned}
\oint_C \frac{\pi\cot(\pi z)\,dz}{(z^2-a^2)(z^2-b^2)} &= \sum_{n=-\infty}^{-1} \frac{-2\pi i}{(n^2-a^2)(n^2-b^2)} - \mathrm{Res}\left[\frac{2\pi^2 i\cot(\pi z)}{(z^2-a^2)(z^2-b^2)}\Bigg|_{-a} + \frac{2\pi^2 i\cot(\pi z)}{(z^2-a^2)(z^2-b^2)}\Bigg|_{-b}\right] \\
&\quad + \sum_{n=1}^{\infty} \frac{-2\pi i}{(n^2-a^2)(n^2-b^2)} - \mathrm{Res}\left[\frac{2\pi^2 i\cot(\pi z)}{(z^2-a^2)(z^2-b^2)}\Bigg|_{a} + \frac{2\pi^2 i\cot(\pi z)}{(z^2-a^2)(z^2-b^2)}\Bigg|_{b}\right] \\
&= -4\pi i \sum_{n=1}^{\infty} \frac{1}{(n^2-a^2)(n^2-b^2)} - 2\pi i \frac{2\pi\cot(\pi a)}{2a(a^2-b^2)} - 2\pi i \frac{2\pi\cot(\pi b)}{(b^2-a^2)2b}
\end{aligned}$$

Evaluating the original contour integral with the residue theorem, the only pole we need consider is that at $z = 0$. No other poles are contained within the full contour, C. Thus we have

$$\begin{aligned}
\oint_C \frac{\pi\cot(\pi z)}{(z^2-a^2)(z^2-b^2)}\,dz &= 2\pi i \,\mathrm{Res}\left(\frac{\pi\cot(\pi z)}{(z^2-a^2)(z^2-b^2)}\right)\Bigg|_{z=0} \\
&= \frac{2\pi i}{a^2 b^2}
\end{aligned}$$

Equating, we find

$$\sum_{n=1}^{\infty} \frac{1}{(n^2-a^2)\,(n^2-b^2)} = -\frac{1}{2a^2\,b^2} + \frac{\pi}{2(a^2-b^2)}\,\left[\frac{1}{b}\cot(\pi b) - \frac{1}{a}\cot(\pi a)\right]$$

Arfken 7.1.1 Find the residues of the following functions.

(a) $1/(z^2+a^2)$ has simple poles at $z = \pm ia$. The residues are

$$\begin{aligned} \frac{1}{z^2+a^2}(z \mp ia)\bigg|_{z=\pm ia} &= \frac{1}{z \pm ia}\bigg|_{z=\pm ia} \\ &= \pm\frac{1}{2ia} \end{aligned}$$

(b) $1/(z^2+a^2)^2$ has second order poles at $z = \pm ia$. The residues are

$$\begin{aligned} \frac{d}{dz}\left(\frac{1}{(z^2+a^2)^2}(z \mp ia)^2\right)\bigg|_{z=\pm ia} &= \frac{-2}{(z \pm ia)^3}\bigg|_{z=\pm ia} \\ &= \pm\frac{1}{4ia^3} \end{aligned}$$

(c) $z^2/(z^2+a^2)^2$ has second order poles at $z = \pm ia$. The residues are

$$\begin{aligned} \frac{d}{dz}\left(\frac{z^2}{(z^2+a^2)^2}(z \mp ia)^2\right)\bigg|_{z=\pm ia} &= \left(\frac{2z}{(z \pm ia)^2} + z^2\frac{-2}{(z \pm ia)^3}\right)\bigg|_{z=\pm ia} \\ &= \mp\frac{i}{4a} \end{aligned}$$

(d) $\sin(1/z)/(z^2+a^2)$ has simple poles at $z = \pm ia$ and an essential singularity at $z = 0$. The residues are

$$\begin{aligned} \frac{\sin(1/z)}{z^2+a^2}(z \mp ia)\bigg|_{z=\pm ia} &= \frac{\sin(1/z)}{z \pm ia}\bigg|_{z=\pm ia} \\ &= \frac{\sinh(1/a)}{2a}. \end{aligned}$$

For $z = 0$ there are no convenient tricks and we must use the full Laurent expansion given by

$$\sin\!\big(\frac{1}{z}\big)\,\frac{1}{z^2+a^2} = \sum_{n=-\infty}^{\infty} a_n z^n$$

where

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\oint_C \sin\!\big(\frac{1}{z}\big)\,\frac{1}{z^2+a^2}\,\frac{dz}{z^{n+1}} \\ &= \frac{1}{2\pi i}\oint \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\big(\frac{1}{z}\big)^(2k+1)\sum_{p=0}^{\infty}\frac{(-1)^p z^{2p}}{a^{2p+2}}\,\frac{dz}{z^{n+1}} \\ &= \sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{k+p}}{(2k+1)!\,a^{2p+2}}\,\frac{1}{2\pi i}\oint z^{2p-2k-n-2}dz \\ &= \sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{k+p}}{(2k+1)!\,a^{2p+2}}\,\delta_{2p-2k-n-2,-1} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^{2k+(n+1)/2}}{(2k+1)!\,a^{2k+n+3}} \\ &= \big(\frac{i}{a}\big)^{n+1}\,\frac{1}{a}\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\,\frac{1}{a^{2k+1}} \\ &= \big(\frac{i}{a}\big)^{n+1}\,\frac{1}{a}\,\sinh\frac{1}{a} \end{aligned}$$

Reading off the residue, a_{-1}, we get $a^{-1}\sinh(a^{-1})$ for the essential singularity at $z = 0$.

(e) $ze^{iz}/(z^2 + a^2)$ has simple poles at $z = \pm ia$ The residues are

$$\left.\frac{ze^{iz}}{z^2 + a^2}(z \mp ia)\right|_{z=\pm ia} = \left.\frac{ze^{iz}}{z \pm ia}\right|_{z=\pm ia}$$
$$= \frac{1}{2}\, e^{\mp a}$$

The point at $z = \infty$ (let $w = 1/z$) is an essential singularity. To get its residue, we must be a bit creative. Transforming to w, our function is

$$f(z) = f(\frac{1}{w}) = g(w) = \frac{we^{i/w}}{1 + a^2w^2}$$

If we ask for the Laurent series of $g(w)$ around $w = 0$, the a_1 term of that series, will be the a_{-1} term of the Laurent series for $f(z)$ around $z = \infty$. So let's find a_1 of $g(w)$:

$$\begin{aligned}
a_1(w = 0) &= \frac{1}{2\pi i} \oint \frac{we^{i/w}}{1 + a^2w^2}\frac{dw}{w^2} \\
&= \frac{1}{2\pi i} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{i^k}{k!}(-a^2)^l \oint w^{-k+2l-1}dw \\
&= \sum_{k=0}^{\infty} \frac{i^k(-1)^{k/2}(a^2)^{k/2}}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} \\
&= e^{-a}
\end{aligned}$$

This, then, is the residue of our original function at the essential singularity at $z = \infty$.

(f) $ze^{iz}/(z^2 - a^2)$ has simple poles at $z = \pm a$ The residues are

$$\left.\frac{ze^{iz}}{z^2 - a^2}(z \mp a)\right|_{z=\pm a} = \left.\frac{ze^{iz}}{z \pm a}\right|_{z=\pm a}$$
$$= \frac{1}{2}\, e^{\pm ia}$$

As in **(e)**, there is an essential singularity as well at $z = \infty$. The only difference here from the previous problem is the sign of the denominator. Applying that here, we have (skipping some steps)

$$\begin{aligned}
a_1(w = 0) &= \sum_{k=0}^{\infty} \frac{i^k(a^2)^{k/2}}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(ia)^k}{k!} \\
&= e^{ia}
\end{aligned}$$

(g) $e^{iz}/(z^2 - a^2)$ has simple poles at $z = \pm a$ The residues are

$$\left.\frac{e^{iz}}{z^2 - a^2}(z \mp a)\right|_{z=\pm a} = \left.\frac{e^{iz}}{z \pm a}\right|_{z=\pm a}$$
$$= \pm\frac{1}{2a}\, e^{\pm ia}$$

Again, we also have an essential singularity at $z = \infty$. Following the two previous examples, we write

$$f(z) = f(\frac{1}{w}) = g(w) = \frac{w^2 e^{i/w}}{1 - a^2 w^2}$$

and we find the $n = 1$ term of the Laurent series of $g(w)$ expanded about $w = 0$ since it corresponds to the a_{-1} term of the Laurent series of $f(z)$ expanded about $z = \infty$:

$$\begin{aligned}
a_1(w=0) &= \frac{1}{2\pi i} \oint \frac{w^2 e^{i/w}}{1 - a^2 w^2} \frac{dw}{w^2} \\
&= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{i^k}{k!} (a^2)^l \oint w^{-k+2l} dw \\
&= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{i^k}{k!} a^{2l} \delta_{2l,k-1} \\
&= \frac{1}{a} \sum_{k=1}^{\infty} \frac{(ia)^k}{k!} \\
&= \frac{1}{a} \left[\sum_{k=0}^{\infty} \frac{(ia)^k}{k!} - 1 \right] \\
&= \frac{1}{a} \left[e^{ia} - 1 \right]
\end{aligned}$$

This, then, is the residue of our original function at the essential singularity at $z = \infty$.

(h) $z^{-k}/(z+1)$ (with $0 < k < 1$) has a simple pole at $z = -1$ and a branch point at $z = 0$. The residue at $z = -1$ is (taking the positive real axis as our cut line – thus $0 < \arg z < 2\pi$)

$$\begin{aligned}
\left. \frac{z^{-k}}{z+1}(z+1) \right|_{z=-1} &= (-1)^{-k} \\
&= (e^{i\pi})^{-k} \\
&= e^{-ik\pi}
\end{aligned}$$

At the branch point at $z = 0$, no residue is defined. The point at $z = \infty$ is also a branch point.

Arfken 7.1.6 Find the generating function for the Bessel and Hermite functions given $g(t,x) = \sum_n f_n(x) t^n$ where $f_n(x)$ is the integral representation of the function.

(a) Bessel function, $J_n(x)$

$$\begin{aligned}
g(t,x) &= \sum_{n=0}^{\infty} t^n \frac{1}{2\pi i} \oint e^{(x/2)(t'-1/t')} t'^{-n-1} dt' \\
&= \sum_{n=0}^{\infty} t^n \frac{1}{2\pi i} \oint \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{x}{2t'}(1-t'^2)\right)^k t'^{-n-1}\, dt' \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} t^n \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{x}{2}\right)^k \oint \sum_{m=0}^{k} \frac{k!}{m!\,(k-m)!} (-t'^2)^m t'^{-k-n-1}\, dt' \\
&= \sum_{n=0}^{\infty} t^n \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{x}{2}\right)^k \sum_{m=0}^{k} \frac{k!}{m!\,(k-m)!} (-1)^m \frac{1}{2\pi i} \oint t'^{2m-k-n-1}\, dt' \\
&= \sum_{n=0}^{\infty} t^n \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{x}{2}\right)^k \sum_{m=0}^{k} \frac{k!}{m!\,(k-m)!} (-1)^m \,\delta_{n,2m-k} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{x}{2}\right)^k \sum_{m=0}^{k} \frac{k!}{m!\,(k-m)!} (-1)^m \, t^{2m-k} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{x}{2t}\right)^k \sum_{m=0}^{k} \frac{k!}{m!\,(k-m)!} (-t^2)^m \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{x}{2t}\right)^k (1-t^2)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x}{2t}(t^2-1)\right)^k \\
&= e^{(x/2)(t-1/t)}
\end{aligned}$$

There is another way to do this problem (as well as part **d**). One can view $f_n(x) = J_n(x)$ as the x-dependent coefficients of a Laurent expansion for the complex (in t) function $g(t,x)$:

$$g(t,x) = \sum_{-\infty}^{\infty} a_n(x)(t-t_0)^n$$

where

$$a_n(x) = \frac{1}{2\pi i} \oint_C \frac{g(t',x)\, dt'}{(t'-t_0)^{n+1}}$$

Because we are given

$$a_n(x) = J_n(x) = \frac{1}{2\pi i} \oint e^{(x/2)(t'-1/t')} t'^{-n-1} dt'$$

if we set $t_0 = 0$, on comparing this with the general form for $a_n(x)$, we can just read off the interior function as $g(t,x)$

$$g(t,x) = e^{(x/2)(t-1/t)}$$

(d) Hermite function, $H_n(x)$ (**NOTE**: There is a mistake in the problem: there should be no $n!$.)

$$
\begin{aligned}
g(t,x) &= \sum_{n=0}^{\infty} t^n \frac{1}{2\pi i} \oint e^{-t'^2+2t'x} t'^{-n-1} dt' \\
&= \sum_{n=0}^{\infty} t^n \frac{e^{x^2}}{2\pi i} \oint e^{-(t'-x)^2} t'^{-n-1} dt' \\
&= \sum_{n=0}^{\infty} t^n \frac{e^{x^2}}{2\pi i} \oint \sum_{k=0}^{\infty} \frac{1}{k!} \left(-(t'-x)^2\right)^k t'^{-n-1} dt' \\
&= \sum_{n=0}^{\infty} t^n \frac{e^{x^2}}{2\pi i} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \oint \sum_{m=0}^{2k} \frac{(2k)!}{m!\,(2k-m)!} t'^{2k-m} (-x)^m t'^{-n-1} dt' \\
&= \sum_{n=0}^{\infty} t^n \frac{e^{x^2}}{2\pi i} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{m=0}^{2k} \frac{(2k)!}{m!\,(2k-m)!} (-x)^m \oint t'^{2k-m-n-1} dt' \\
&= \sum_{n=0}^{\infty} t^n e^{x^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{m=0}^{2k} \frac{(2k)!}{m!\,(2k-m)!} (-x)^m \, \delta_{n,2k-m} \\
&= e^{x^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{m=0}^{2k} \frac{(2k)!}{m!\,(2k-m)!} t^{2k-m} (-x)^m \\
&= e^{x^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (t-x)^{2k} \quad = e^{-t^2+2tx}
\end{aligned}
$$

Arfken 7.1.14

(a) Evaluate $I_{\mathrm{a}} = \int_{-\infty}^{\infty} \cos x/(x^2+a^2)\,dx$ for $a > 0$. First, note that $\cos z$ in the corresponding contour integral will diverge for a contour with semi-circle that closes in either the upper or lower half planes. The trick is to do $\int_{-\infty}^{\infty} e^{ix}/(x^2+a^2)\,dx$ and take only the real part at the end. If our contour is the real axis and a semi-circle in the upper half plane, the integral along the semi-circle will go to zero by Jordan's Lemma and we need only evaluate the residue at $z = ia$. thus

$$
\begin{aligned}
\oint \frac{e^{iz}}{z^2+a^2}\,dz &= \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\,dx \\
&= 2\pi i \frac{e^{i(ia)}}{2ia} \\
&= \frac{\pi}{a} e^{-a}
\end{aligned}
$$

This is real, so I_{a} equals this. If we change things to include $\cos kx$, we have to be more careful how we close the contour. Doing the same trick, and considering e^{ikz} in the contour integral, note that the exponential can be written $e^{ikr\cos\theta - kr\sin\theta}$ and will decay (and the corresponding integral in θ) in the upper half plane if $k > 0$. If $k < 0$, then decay will occur only if we close the contour in the lower half plane. The latter case means that we would encircle the pole at $z = -ia$ instead of $z = ia$. This would give

$$
\begin{aligned}
\oint \frac{e^{ikz}}{z^2+a^2}\,dz &= \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+a^2}\,dx \\
&= \begin{cases} 2\pi i \frac{e^{i(ika)}}{2ia} & k > 0 \\ -2\pi i \frac{e^{i(-ika)}}{-2ia} & k < 0 \end{cases} \\
&= \frac{\pi}{a} e^{-|k|a}
\end{aligned}
$$

(b) Evaluate $I_\mathrm{b} = \int_{-\infty}^{\infty} x \sin x/(x^2+a^2)\,dx$ for $a > 0$. Taking the same approach as in part (a), do $\int_{-\infty}^{\infty} xe^{ix}/(x^2+a^2)\,dx$ and take only the imaginary part at the end of the calculation. If our contour is the real axis and a semi-circle in the upper half plane, the integral along the semi-circle will go to zero by Jordan's Lemma and we need only evaluate the residue at $z = ia$. thus

$$\begin{aligned}\oint \frac{ze^{iz}}{z^2+a^2}\,dz &= \int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+a^2}\,dx \\ &= 2\pi i \frac{iae^{i(ia)}}{2ia} \\ &= i\pi e^{-a}\end{aligned}$$

This is imaginary, so $I_\mathrm{b} = \pi e^{-a}$. If we change things to include $\sin kx$, we follow the same reasoning as before and close our contour in the upper half plane if $k > 0$ and for $k < 0$, we close our contour in the lower half plane. Again, the latter case means that we calculate the residue at $z = -ia$. Putting it all together, we get

$$\begin{aligned}\oint \frac{ze^{ikz}}{z^2+a^2}\,dz &= \int_{-\infty}^{\infty} \frac{xe^{ikx}}{x^2+a^2}\,dx \\ &= \begin{cases} 2\pi i \frac{iae^{i(ika)}}{2ia} & k > 0 \\ -2\pi i \frac{-iae^{i(-ika)}}{-2ia} & k < 0 \end{cases} \\ &= i\pi e^{-|k|a} \begin{cases} 1 & k > 0 \\ -1 & k < 0 \end{cases}\end{aligned}$$

Since this is imaginary, we drop the i, and the result is what we want for I_b.

Arfken 7.1.17 Evaluate $I = \int_0^{\infty} (\ln x)^2/(1+x^2)\,dx$

Let $x = e^y$ and the integral becomes

$$\begin{aligned}I &= \int_{-\infty}^{\infty} \frac{y^2 e^y dy}{1+e^{2y}} \\ &= \int_{-\infty}^{\infty} \frac{y^2 dy}{e^{-y}+e^y}\end{aligned}$$

which, because the denominator is $2\cosh y$ demonstrates the integrand is an even function of y and allows us to take the limits from 0 to ∞ and multiply by 2. This leads to a simpler expansion

$$\begin{aligned}I &= 2\int_0^{\infty} \frac{y^2 dy}{e^{-y}+e^y} \\ &= 2\int_0^{\infty} \frac{y^2 e^{-y}}{1+e^{-2y}}\,dy \\ &= 2\int_0^{\infty} y^2 e^{-y} \sum_{n=0}^{\infty} (-1)^n e^{-2ny}\,dy \\ &= 2\sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} y^2 e^{-y(2n+1)}\,dy \\ &= 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \int_0^{\infty} w^2 e^{-w}\,dw \qquad \text{where} \quad w = y(2n+1) \\ &= 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}\, 2! \\ &= 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}\end{aligned}$$

as desired.

(b) Now, using contour integration, we take

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2 dx}{\cosh x}$$

as the integral we want. In the z plane the contour integral should be this with $x \to z$ and the contour we want to use is a rectangle lying on the real axis with top at $y = i\pi$. This way, we enclose a simple pole at $y = i\pi/2$. Note that, as in some earlier problems, the integrals along the vertical sides go to zero, so we won't consider them further. Our contour integral becomes

$$\begin{aligned}\oint_C \frac{z^2}{2\cosh z} &= I + \int_{\infty}^{-\infty} \frac{(x+i\pi)^2}{2\cosh(x+i\pi)}\,dx \\ &= I + \int_{-\infty}^{\infty} \frac{x^2 + 2ix\pi - \pi^2}{2\cosh x}\,dx\end{aligned}$$

where we have expanded $\cosh(x + i\pi)$ and used $\cosh(i\pi) = -1$ and $\sinh(i\pi) = 0$. Note that the middle term in the numerator goes to zero because the integral is of an odd function over a symmetric interval. The last term in the denominator gives another integral. We could do this by another contour integration, but to save on work we can also recall that this is just the integral of $1/\cosh x$ which is $2\arctan(\sinh x)$ evaluated between 0 and ∞. This yields π. So we have

$$\oint_C \frac{z^2}{2\cosh z} = 2I - \frac{\pi^3}{2}$$

from the residue theorem the contour integral is

$$\begin{aligned}\oint_C \frac{z^2}{2\cosh z} &= 2\pi i \left.\frac{z^2}{2\sinh z}\right|_{z=i\pi/2} \\ &= -\frac{\pi^3}{4}\end{aligned}$$

giving, finally,

$$I = \int_0^{\infty} \frac{x^2 dx}{\cosh x} = \frac{\pi^3}{8}$$

Arfken 7.1.19 Evaluate $I = \int_0^\infty x^{-a}/(x+1)\,dx$ where $0 < a < 1$.

We will have a branch point at $z = 0$ so we will use the positive x-axis as the branch cut. There will be a pole at $z = -1$ of order 1. The contour we will take will include (1) a horizontal line just above the x-axis from 0 to ∞, (2) a circle with radius R that will eventually $\to \infty$, (3) a horizontal line from ∞ to 0 just below the positive real axis, and (4) a tiny circle of radius ϵ around the branch point at $z = 0$. The contour integral thus becomes

$$\begin{aligned}\oint_C \frac{z^{-a}}{z+1}\,dz = \lim_{R\to\infty}\lim_{\epsilon\to 0} \Bigg\{ &\int_\epsilon^R \frac{x^{-a}}{x+1}\,dx + \int_0^{2\pi} \frac{(Re^{i\theta})^{-a}}{Re^{i\theta}+1}\, Re^{i\theta}\, id\theta \\ &+ \int_R^\epsilon \frac{(re^{i2\pi})^{-a}}{re^{i2\pi}+1}\, dre^{i2\pi} + \int_{2\pi}^0 \frac{(\epsilon e^{i\theta})^{-a}}{\epsilon e^{i\theta}+1}\, \epsilon e^{i\theta} id\theta \Bigg\} \\ = I\big(1 - e^{-i2a\pi}\big)&\end{aligned}$$

Note that the angular integral with R large only vanishes if $a > 0$. The other angular integral around the small circle of radius ϵ vanishes provided $a < 1$. Hence, we have the bounds on a as stated in the problem. The residue theorem gives, for the contour integral,

$$\oint_C \frac{z^{-a}}{z+1}\, dz = 2\pi i\, (-1)^{-a}$$
$$= 2\pi i\, e^{-i\pi a}$$

Putting this together, we get

$$I = \int_0^\infty \frac{x^{-a}}{x+1}\, dx = \frac{\pi}{\sin \pi a}$$