Assignment 9

(1) Evaluate the asymptotic dependance of $I(x) = \int_0^\infty e^{x(t-e^t)}\,dt$ for $x >> 1$.

Note first that the integrand goes to zero as $t \to \infty$ since e^t will dominate t. However, the integrand goes to e^{-x} as $t \to 0$. This could potentially be bad as the assumptions on our integral are that the integrand must go to zero at the limits of integration. Otherwise steepest descents may not work. Proceeding naively for the moment, the function $f(t) = t - e^t$ has saddle points at $t_0 = 0$, thus the lower limit of integration ends on the saddle point. If our integrand were even, we could simply extend the integral to $-\infty$ and divide by two, but here we have to be more careful. However, since the major contribution to the asymptotic dependance will come from the vicinity of the saddle point, we can make the following argument. Near $t_0 = 0$, the function can be expanded to be $t - (1 + t + t^2/2 + \cdots) \approx 1 - t^2/2$ which is even. Now, since the saddle point and its vicinity give the integral its value, we will say that in the region of main interest, the integrand is effectively even extend our integration to $-\infty$ and then divide by two.

The phase of $f''(t) = -e^t$ at the saddle point is $\beta = \pi$ thus $\alpha = 0$. Putting it all together,

$$\begin{aligned} I(x) &\sim \frac{1}{2}\,\frac{\sqrt{2\pi}\,e^{-x}}{|x(-1)|^{1/2}} \\ &= \sqrt{\frac{\pi}{2x}}\,e^{-x} \end{aligned}$$

(2) Evaluate the asymptotic dependance of $I(x) = \int_0^\infty t^2 e^{t^2 - xt^4}\,dt$ for $x << 1$.

Note that the integral goes to zero in both integration limits. To make use of steepest descents, we need a large parameter, so we will use $1/x$. Factoring it out of the exponential we have

$$I(x) = \int_0^\infty t^2 e^{\frac{1}{x}(xt^2 - x^2t^4)}\,dt$$

and can now make the substitution $u^2 = xt^2$. (Note, you could also substitute $u = xt^2$ and though the subsequent details will be a bit different, the answer will be exactly the same.) This transforms the integral to

$$I(x) = \frac{1}{x^{3/2}} \int_0^\infty u^2 e^{\frac{1}{x}(u^2 - u^4)}\,du$$

The function $f(u) = u^2 - u^4$ has saddle points at $u = 0, \pm 1/\sqrt{2}$. Thus an endpoint of our integral sits on one of the saddle points. In this case, we can extend the integration to $-\infty$ without worry since the integrand is even. However, with three saddle points, we need to consider, in general, the contribution to the integral from each. We will denote them as u_0 and $u_\pm$. The second derivative of $f(u)$ is $f''(u) = 2 - 12u^2$ with the phase, β_0, of $f''(0) = 2$ being 0. The phases, $\beta_\pm$, of $f''(\pm 1/\sqrt{2}) = -4$ are both π. In addition, $f(0) = 0$, $f(\pm 1/\sqrt{2}) = 1/4$, $g(0) = 0$ and $g(\pm 1/\sqrt{2}) = 1/2$.

Putting it all together, we have

$$\begin{aligned} I(x) &\sim \frac{1}{2}\,\frac{1}{x^{3/2}} \left[\frac{\sqrt{2\pi}(1/2)e^{(1/4x)}}{|\frac{1}{x}(-4)|^{1/2}} + \frac{\sqrt{2\pi}(0)e^{(0\cdot x)}e^{i\pi/2}}{|\frac{1}{x}2|^{1/2}} + \frac{\sqrt{2\pi}(1/2)e^{(1/4x)}}{|\frac{1}{x}(-4)|^{1/2}} \right] \\ &= \sqrt{2\pi}\,\frac{e^{1/4x}}{4x} \end{aligned}$$

Note that $g(0) = 0$ actually violates one of our assumptions regarding the use of steepest descents – namely that $g(t_0)$ should not be zero near the saddle point. However, in this case, one can check that the contribution from the saddle point at $u_0 = 0$ is indeed negligible in higher orders of the asymptotic series thus verifying that this is the leading order contribution.

Arfken 7.2.1

A function $f(z)$ satisfies the conditions for the dispersion relations as well as the Schwarz reflection principle: $f(z) = f^*(z^*)$. Begin with the same derivation of the dispersion relations as in the text. We assume that $\lim_{|z|\to\infty} |f(z)| = 0$ in the upper half plane. Therefore, Cauchy's integral formula (or similarly the residue theorem) gives, for any point z_0 in the upper half plane

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x - z_0} dx \end{aligned}$$

where we have closed C in the upper half plane and hence that part of the contour integral $\to 0$. Note that our results gives us the value of f anywhere in the upper half plane via an integral along only the real axis. At this point, the "usual" dispersion relations were derived by letting z_0 move onto the real axis, *i.e.* become real. Don't do that here. Instead, let's derive new dispersion relations from the above by just taking real and imaginary parts

$$\begin{aligned} u(x_0, y_0) + iv(x_0, y_0) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{u(x,0) + iv(x,0)\big)(x - x_0 + iy_0)}{(x - x_0 - iy_0)(x - x_0 + iy_0)} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{y_0 u(x,0) + (x - x_0)v(x,0)}{(x - x_0)^2 + y_0^2} - i \frac{(x - x_0)u(x,0) - y_0 v(x,0)}{(x - x_0)^2 + y_0^2} dx \end{aligned}$$

The Schwarz reflection principle is nothing more than a symmetry statement about the real and imaginary parts of $f(z)$. In particular, it states $u(x,y) + iv(x,y) = u(x,-y) - iv(x,-y)$, or that $u(x,y)$ is an even function of y and that $v(x,y)$ is an odd functions of y. As such, $v(x,0) = 0$, and terms above containing it go to zero. Then we have

$$\begin{aligned} u(x_0, y_0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{y_0 u(x,0)}{(x - x_0)^2 + y_0^2} dx \\ v(x_0, y_0) &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(x - x_0)u(x,0)}{(x - x_0)^2 + y_0^2} dx \end{aligned}$$

Now, since u is even with respect to y, it must satisfy $u(x_0, y_0) = u(x_0, -y_0)$ and we have

$$u(x_0, -y_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-y_0 u(x,0)}{(x - x_0)^2 + (-y_0)^2} dx$$

but this will only be true provided $u(x_0, y_0) = 0$. Similarly, v being odd with respect to y, it must satisfy $v(x_0, y_0) = -v(x_0, -y_0)$ or

$$-v(x_0, -y_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(x - x_0)u(x,0)}{(x - x_0)^2 + (-y_0)^2} dx$$

which, again, will only be true if $v(x_0, y_0) = 0$. Since both real and imaginary parts of $f(z_0)$ are zero, it is identically zero.

Arfken 7.2.4

Find the asymptotic behavior of the dispersion relations with $f(x) = f^*(-x)$ symmetry. The dispersion relations are

$$u(x_0) = \frac{2}{\pi} P \int_0^\infty \frac{x v(x)}{x^2 - x_0^2}\, dx$$
$$v(x_0) = -\frac{2}{\pi} P \int_0^\infty \frac{x_0 u(x)}{x^2 - x_0^2}\, dx$$

In both we can expand the denominator for large x_0 as

$$\begin{aligned}\frac{1}{x^2 - x_0^2} &= -\frac{1}{x_0^2} \frac{1}{1 - (x/x_0)^2} \\ &= -\frac{1}{x_0^2} \left(1 + \frac{x^2}{x_0^2} + \frac{x^4}{x_0^4} + \cdots\right)\end{aligned}$$

Note that this expansion will be strictly valid only for $x < x_0$ while the integral actually extends to ∞ in x. Thus, there will be a region where the expansion will not converge. This makes the resulting series asymptotic rather than convergent. Provided x_0 is large enough the contributions from the nonconvergent region will be taken as negligible. Taking this expansion and keeping only the first terms in each yields

$$u(x_0) \sim -\frac{2}{\pi x_0^2} P \int_0^\infty x v(x)\, dx$$
$$v(x_0) \sim \frac{2}{\pi x_0} P \int_0^\infty u(x)\, dx$$

Arfken 7.2.5

Using the dispersion relations (Hilbert transforms) with the integral equation

$$\frac{1}{1 + x_0^2} = \frac{1}{\pi} P \int_{-\infty}^\infty \frac{u(x)}{x - x_0}\, dx$$

allows us to identify $-v(x_0)$ (the imaginary part of a function) with the left hand side. Using the second Hilbert transform (or solution to the real part), we have

$$\begin{aligned}u(x_0) &= \frac{1}{\pi} P \int_{-\infty}^\infty \frac{v(x)}{x - x_0}\, dx \\ &= -\frac{1}{\pi} P \int_{-\infty}^\infty \frac{dx}{(1 + x^2)(x - x_0)}\end{aligned}$$

For our purposes, evaluating the principal value equates to using the residue theorem with simple poles on the contour contributing half their normal value. We will do the above integral via contour methods by evaluating

$$\oint_C \frac{dz}{(1 + z^2)(z - z_0)}$$

with C running along the real axis and closed as a large semi-circle in the upper half plane. The angular part goes to zero and the real part is the integral we want. So

$$\begin{aligned}u(x, 0) &= -\frac{1}{\pi} 2\pi i \left(\frac{1}{2z} \frac{1}{z - x_0} \bigg|_{z=i} + \frac{1}{2} \frac{1}{1 + z^2} \bigg|_{z=x_0} \right) \\ &= \frac{x_0}{1 + x_0^2}\end{aligned}$$

(b) Substituting this back into the integral equation, we have (using the same contour as in (a))

$$\begin{aligned}
\frac{1}{1+x_0^2} &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{x}{1+x^2}\,\frac{1}{x-x_0}\,dx \\
&= \frac{1}{\pi} P \oint_C \frac{z}{1+z^2}\,\frac{1}{z-x_0}\,dz \\
&= 2i\left(\frac{z}{2z}\,\frac{1}{z-x_0}\bigg|_{z=i} + \frac{1}{2}\,\frac{z}{1+z^2}\bigg|_{z=x_0} \right) \\
&= \frac{1}{1+x_0^2}
\end{aligned}$$

which checks out.

(c) Since $f(z)|_{y=0} = u(x) + iv(x)$, we can let $x \to z$ and get

$$\begin{aligned}
f(z) &= \frac{z}{1+z^2} - i\,\frac{1}{1+z^2} \\
&= \frac{1}{z+i}
\end{aligned}$$

(d) The crossing conditions are the symmetry relations, $u(-x) + iv(-x) = u(x) - iv(x)$. Since here, $u(x)$ is odd and $v(x)$ is even, the crossing conditions are *not* satisfied.

Arfken 7.2.6

The Kronig-Kramers optical dispersion relations are

$$\begin{aligned}
\Re[n^2(\omega_0) - 1] &= \frac{2}{\pi} P \int_0^{\infty} \frac{\omega \Im[n^2(\omega) - 1]}{\omega^2 - \omega_0^2}\,d\omega \\
\Im[n^2(\omega_0) - 1] &= -\frac{2}{\pi} P \int_0^{\infty} \frac{\omega_0 \Re[n^2(\omega) - 1]}{\omega^2 - \omega_0^2}\,d\omega
\end{aligned}$$

If, in the second equation, the real part $n^2 - 1$ is a constant, that can be pulled out of the integral and we have

$$\begin{aligned}
\Im[n^2(\omega_0) - 1] &= -\frac{2\omega}{\pi} \Re[n^2 - 1]\, P \int_0^{\infty} \frac{1}{\omega^2 - \omega_0^2}\,d\omega \\
&= -\frac{\omega}{\pi} \Re[n^2 - 1] \oint_C \frac{1}{z^2 - \omega_0^2}\,dz \\
&= -\frac{2\omega}{\pi} \Re[n^2 - 1]\,\frac{1}{2}\,2\pi i \left[\frac{1}{2z}\bigg|_{z=-\omega_0} + \frac{1}{2z}\bigg|_{z=\omega_0} \right] \\
&= 0
\end{aligned}$$

and the imaginary part (absorptive part) is zero. Note that in the second line, before going to the contour integral, we extend the limits of integration to include the entire real axis as the integrand is an even function of ω.

(b) Now assume that the absorptive part of the index of refraction is not zero. We could make the assumption that the absorptive part is constant, but we can make our argument more general. Note that the above form of the optical dispersion relations require that the real part of $n^2 - 1$ be an even function of ω and the imaginary part be an odd function. The integrand for the calculation of the real part is now an even function, with poles on the real axis. Turning this into a contour integral, we need to consider the imaginary part as a function itself of the complex variable z, but it is *not* necessarily analytic. Recall, it was $n^2(\omega) - 1$ which was assumed analytic in the upper half ω plane, but we have no such condition on the imaginary part of n^2-1 as a separate function of some new complex variable z. Indeed, if $\Im(n^2(z)-1)$ is not to be a constant, it will certainly have poles in the z plane. However, we are assured that it $\to 0$ as $|z|\to\infty$. Thus we can write

$$\begin{aligned}
\Re[n^2(\omega_0)-1] &= \frac{2}{\pi}P\int_0^\infty \frac{\omega\Im[n^2(\omega)-1]}{\omega^2-\omega_0^2}\,d\omega \\
&= \frac{1}{\pi}\oint_C \frac{z\Im[n^2(z)-1]}{z^2-\omega_0^2}\,dz \\
&= \frac{1}{\pi}\frac{1}{2}2\pi i\left(\left.\frac{z\Im[n^2(z)-1]}{2z}\right|_{z=-\omega_0}+\left.\frac{z\Im[n^2(z)-1]}{2z}\right|_{z=\omega_0}\right)+\frac{1}{\pi}2\pi i\sum_{\text{poles}}\text{Res}\,\frac{z\Im[n^2(z)-1]}{z^2-\omega_0^2} \\
&= 2i\sum_{\text{poles}}\text{Res}\,\frac{z\Im[n^2(z)-1]}{z^2-\omega_0^2}
\end{aligned}$$

where the sum over the "poles" is *only* of the poles of the numerator. The poles at $z=\pm\omega_0$ are taken care of and in fact cancel since $\Im(n^2(z)-1)$ is odd.

Arfken 7.3.1. Evaluate, using steepest descents, the second Hankel function.

The contour that makes the integrand $\to 0$ at both limits is one that comes in from $-\infty$ (just below a branch cut from $z=0$ to $z=-\infty$), encircles the origin in a positive sense and then extends out to $z=-\infty$ (now just above the branch cut). We then have

$$H_\nu^{(2)}(s)=\frac{1}{\pi i}\int_C e^{(s/2)(z-1/z)}\,\frac{dz}{z^{\nu+1}}$$

The function is $f(z)=(z-1/z)/2$. The locations of its saddle points are at $z_0=\pm i=e^{\pm i\pi/2}$ and $f''(z_0)=-z_0^{-3}=e^{\mp i\pi/2}$. We will take the $z_0=-i$ saddle point as the other saddle point contributes to the first Hankel function. Thus the phase of the second derivative, $\beta=\pi/2$ and the phase of the path through the contour is $\alpha=(\pi-\beta)/2=\pi/4$. Putting all this together, we can use our formula

$$\begin{aligned}
H_\nu^{(2)}(s) &\approx \frac{1}{\pi i}\frac{\sqrt{2\pi}g(z_0)e^{sf(z_0)}e^{i\alpha}}{\sqrt{|sf''(z_0)|}} \\
&= \sqrt{\frac{2}{\pi s}}\,\frac{1}{(-i)^\nu}\,e^{-is}e^{i\pi/4} \\
&= \sqrt{\frac{2}{\pi s}}\,e^{-i(s-\nu\pi/2-\pi/4)}
\end{aligned}$$

Arfken 7.3.4 Evaluate the asymptotic dependence of the modified Bessel functions

$$I_\nu(x)=\frac{1}{2\pi i}\int_C e^{(x/2)(t+1/t)}\,\frac{dt}{t^{\nu+1}}$$

with a contour, C, which starts at $-\infty$, (just below a branch cut from $z=0$ to $z=-\infty$), encircles the origin in a positive sense and then extends out to $z=-\infty$ (now just above the branch cut). It is straightforward to verify that the integrand goes to zero in both of these limits. The function $f(t)=(t+1/t)/2$ has saddle points at $t=\pm1$. With a branch cut along the negative x-axis, we need only include the saddle point at $t=1$. So $f(1)=1$ and $f''(1)=1$ with phase $\beta=0$. The phase of the contour through the saddle point is $\alpha=\pi/2$. Putting it together, we have

$$\begin{aligned}
I_\nu(x) &\approx \frac{1}{2\pi i}\frac{\sqrt{2\pi}e^x e^{i\pi/2}}{\sqrt{x}} \\
&= \frac{e^x}{\sqrt{2\pi x}}
\end{aligned}$$

Arfken 7.3.5 Evaluate the asymptotic dependence of the modified Bessel function of the second kind,

$$K_\nu(x) = \frac{1}{2}\int_0^\infty e^{(-x/2)(s+1/s)}\,\frac{ds}{s^{1-\nu}}$$

This is a real integral but letting s become complex, we can deform the original path from the positive real axis into the complex plane in order to pass through any saddle points to give us the asymptotic expansion for the integral. In general, there will be a branch point at $s = 0$ for noninteger ν and a branch cut that extends from $s = 0$ to $s = \infty$ which we will take to be along the negative real axis. The saddle points of the function $f(s) = -(s + 1/s)/2$ are at $s_0 = \pm 1$. We need consider only $s = 1$. The second derivative at the saddle point is $f''(1) = -1$ with phase $\beta = \pi$. The phase of the contour through the saddle point is 0 and $f(1) = -1$. Putting it all together,

$$K_\nu(x) \approx \frac{1}{2}\frac{\sqrt{2\pi}e^{-x}}{\sqrt{x}}$$