Midterm – Key

1. **(a)** Prove the identity

$$\vec{u} \times \big(\vec{v} \times \vec{w}\big) + \vec{v} \times \big(\vec{w} \times \vec{u}\big) + \vec{w} \times \big(\vec{u} \times \vec{v}\big) = 0$$

(b) Show that for any *arbitrary* vector field $\vec{u}$ and scalar field ϕ

$$\oint_S \vec{u} \times \vec{\nabla}\phi \;\cdot \hat{n}\, da = \int_V \vec{\nabla}\phi \cdot \big(\vec{\nabla} \times \vec{u}\big)\, dv$$

where V is an arbitrary volume bounded by the surface S.

(a) In index notation, we have

$$\begin{aligned}
&\epsilon_{ijk} u_j \big(\epsilon_{klm} v_l w_m\big) + \epsilon_{ijk} v_j \big(\epsilon_{klm} w_l u_m\big) \\
&\qquad + \epsilon_{ijk} w_j \big(\epsilon_{klm} u_l v_m\big) = \big(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\big)\big(u_j v_l w_m + v_j w_l u_m + w_j u_l v_m\big) \\
&\qquad\qquad = v_i u_j w_j - w_i u_j v_j + w_i v_j u_j - u_i v_j w_j + u_i w_j v_j - v_i w_j u_j \\
&\qquad\qquad = 0
\end{aligned}$$

(b) The divergence theorem for an arbitrary vector $\vec{A}$ is

$$\int_V \vec{\nabla} \cdot \vec{A}\, dv = \oint_S \vec{A} \cdot \hat{n}\, da$$

Using the divergence theorem with $\vec{A} = \vec{u} \times \vec{\nabla}\phi$ for the arbitrary vector $\vec{u}$ and scalar ϕ, we can write

$$\begin{aligned}
\oint_S \vec{u} \times \vec{\nabla}\phi \;\cdot \hat{n}\, da &= \int_V \vec{\nabla} \cdot \big(\vec{u} \times \vec{\nabla}\phi\big)\, dv \\
&= \int_V \Big[\vec{\nabla}\phi \cdot \big(\vec{\nabla} \times \vec{u}\big) - \vec{u} \cdot \big(\vec{\nabla} \times \vec{\nabla}\phi\big)\Big]\, dv \\
&= \int_V \vec{\nabla}\phi \cdot \big(\vec{\nabla} \times \vec{u}\big)\, dv
\end{aligned}$$

where the second term in the second line is zero since the curl of any gradient always vanishes.

2. Show that the following function is regular (finite) as $x \to 0$ and find the first nonzero term in the expansion

$$f(x) = \left(\frac{15}{x^4} - \frac{6}{x^2}\right)\sin x - \left(\frac{15}{x^3} - \frac{1}{x}\right)\cos x$$

Express the trig functions in terms of their infinite series:

$$\begin{aligned}
f(x) &= \left(\frac{15}{x^4} - \frac{6}{x^2}\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) - \left(\frac{15}{x^3} - \frac{1}{x}\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) \\
&= \frac{1}{x^3}(15 - 15) + \frac{1}{x}\left[-\frac{15}{3!} - 6 + \frac{15}{2!} + 1\right] + x\left[\frac{15}{5!} + \frac{6}{3!} - \frac{15}{4!} - \frac{1}{2!}\right] + x^3\left[-\frac{15}{7!} - \frac{6}{5!} + \frac{15}{6!} + \frac{1}{4!}\right] + \cdots \\
&= \frac{x^3}{3 \cdot 5 \cdot 7} + \cdots
\end{aligned}$$

Therefore, as $x \to 0$, $f(x) \to 0$ with the leading order term in its series being the x^3 term above.

3. Find the Laurent expansions around $z = i$ and $z = 1$ of

$$f(z) = \frac{1}{z^2+1}$$

and determine the regions in the complex plane where they are valid.

We want to find the coefficients of the Laurent series

$$f(x) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

where $z_0 = i$ and a_n is

$$\begin{aligned}
a_n &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \\
&= \frac{1}{2\pi i} \oint_C \frac{1}{(z-i)^{n+2}} \frac{dz}{z+i} \\
&= \frac{1}{2\pi i} \oint_C \frac{1}{(z-i)^{n+2}} \frac{dz}{2i+(z-i)} \\
&= \frac{1}{2\pi i} \oint_C \frac{1}{(z-i)^{n+2}} \frac{1}{2i} \sum_{k=0}^{\infty} \frac{(-1)^k (z-i)^k}{(2i)^k}\, dz \\
&= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2i)^{k+1}} \oint_C \frac{dz}{(z-i)^{n+2-k}}
\end{aligned}$$

Letting $r\, e^{i\theta} = z - i$ and choosing a circular contour of radius $r = \epsilon$ around $z = i$, we have

$$\begin{aligned}
a_n &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2i)^{k+1}} \int_0^{2\pi} \frac{\epsilon i e^{i\theta} d\theta}{\epsilon^{n+2-k} e^{i\theta(n+2-k)}} \\
&= \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2i)^{k+1}}\, \epsilon^{-n-1+k} \int_0^{2\pi} e^{i\theta(-n-1+k)} d\theta \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2i)^{k+1}}\, \delta_{k,n+1} \\
&= \begin{cases} 0 & n < -1 \\ \frac{(-1)^{n+1}}{(2i)^{n+2}} & n \geq -1 \end{cases}
\end{aligned}$$

so that

$$f(z) = -\sum_{n=-1}^{\infty} \frac{1}{(2i)^{n+2}} (i-z)^n$$

Or, if you want a simpler method, try

$$\begin{aligned}
\frac{1}{z^2+1} &= \frac{1}{z-i} \cdot \frac{1}{z+i} \\
&= \frac{1}{z-i} \frac{1}{2i+(z-i)} \\
&= \frac{1}{z-i} \frac{1}{2i} \frac{1}{1+(z-i)/(2i)} \\
&= \frac{1}{2i(z-i)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2i)^k} (z-i)^k \\
&= -\sum_{k=0}^{\infty} \frac{1}{(2i)^{k+1}} (i-z)^{k-1} \\
&= -\sum_{n=-1}^{\infty} \frac{1}{(2i)^{n+2}} (i-z)^n
\end{aligned}$$

For the expansion around $z = 1$, we note immediately that the point $z = 1$ is a point of analyticity of the function $f(z)$ and so our Laurent series is, in fact, a Taylor series. So performing the first few derivatives, we get

$$f'(z) = -\frac{2z}{(z^2+1)^2}$$
$$f''(z) = \frac{6z^2-2}{(z^2+1)^3}$$
$$f'''(z) = \frac{-24z^3+24z}{(z^2+1)^4}$$

So we get

$$f(z) = \frac{1}{2} - \frac{1}{2}(z-1) + \frac{1}{4}(z-1)^2 + \mathcal{O}(z-1)^4$$

4. Determine the convergence or divergence of the following infinite series (p and a are constants)

$$\textbf{(a)} \quad \sum_{n=2}^{\infty} \frac{(-1)^n\, n}{\sqrt{n^3-1}} \qquad \textbf{(b)} \quad \sum_{n=1}^{\infty} \frac{1}{n^p\, a^n}$$

(c) Find allowable values of k, p, a and b so that the series below converges. k is an even integer (≥ 0), p is an odd integer (≥ -1), and a and b are positive, real numbers.

$$\sum_{n=0}^{\infty} \frac{(2n+p)!!}{(2n+k)!!}\,(a\,n+b)$$

(a) One way to do this problem is to notice that for large n, the individual terms in the series behave like $n^{-1/2}$ (ignoring the $(-1)^n$). This suggests that this is even more divergent than the harmonic series. So let's try a comparison test. For large n,

$$\begin{aligned} n^3 - 1 &< n^4 \\ \sqrt{n^3-1} &< n^2 \\ \frac{1}{n} &< \frac{n}{\sqrt{n^3-1}} \end{aligned}$$

and we conclude that

$$\sum_{n=2}^{\infty} \frac{1}{n} < \sum_{n=2}^{\infty} \frac{n}{\sqrt{n^3-1}}$$

and our sum diverges, or is not absolutely convergent. As $n \to \infty$, since the terms in the series, $u_n \to 0$, it turns out that we satisfy the Leibnitz criteria and we have conditional convergence, but again, not absolute convergence.

(b) Using the ratio test we have

$$\begin{aligned} \frac{u_n}{u_{n+1}} &= \frac{(n+1)^p\,|a|^{n+1}}{n^p\,|a|^n} \\ &= \left(\frac{n+1}{n}\right)^p |a| \end{aligned}$$

which, as $n \to \infty$ becomes $|a|$. So, if $|a| > 1$, we have absolute convergence, and if $|a| < 1$, the series diverges. We must treat $|a| = 1$ separately. But for $a = 1$, the series is just $\zeta(p)$, the Riemann zeta function which we know converges for $p > 1$ and diverges for $p \leq 1$. The same holds for $a = -1$, but now there is also conditional convergence if $0 < p \leq 1$. Summarizing, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^p\, a^n} \quad \begin{cases} \text{converges absolutely} & \text{if } |a| > 1 \text{ or } |a| = 1 \text{ and } p > 1 \\ \text{diverges} & \text{if } |a| < 1 \text{ or } |a| = 1 \text{ and } p \leq 1 \\ \text{converges conditionally} & \text{if } a = -1 \text{ and } 0 < p \leq 1 \end{cases}$$

(c) Using Gauss' test, we have

$$\begin{aligned} \frac{u_n}{u_{n+1}} &= \frac{(2n+2+k)\,(an+b)}{(2n+2+p)\,(an+a+b)} \\ &= \frac{n^2 + n\cdot[a(2+k)+2b]\,/(2a) + (2+k)/(2a)}{n^2 + n\cdot[2(a+b)+a(2+p)]\,/(2a) + (a+b)(2+p)/(2a)} \end{aligned}$$

The condition from Gauss' test is that the coefficient of n on the top be at least one greater than the coefficient of n on the bottom: $a_1 > b_1 + 1$. In our case this is

$$\frac{1}{2a}\,(a(2+k)+2b) > \frac{1}{2a}\,(2(a+b)+a(2+p)) + 1$$

which simplifies to $k > p + 4$. So, for absolute convergence, we must have $k > p + 4$, otherwise the series diverges. Note that a and b are completely arbitrary (other than being positive as stated in the problem).

5. **(a)** Find the imaginary part, $v(x, y)$ of an analytic function, $f(z)$, if its real part is given by

$$u(x, y) = e^x \left(y \cos y + x \sin y\right)$$

(b) For any second or third order polynomial, $p(z)$, with simple zeros in the complex plane, show

$$\oint_C \frac{1}{p(z)}\, dz = 0$$

where C encloses all the zeros of $p(z)$.

(a) If $f(z)$ is analytic, then the Cauchy-Riemann conditions are satisfied

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and we can use these to solve for $v(x, y)$. The first equation says

$$\begin{aligned} v(x, y) &= \int e^x \left[y \cos y + (x + 1) \sin y\right] dy \\ &= e^x \left[y\ \sin y + \cos y - (x + 1) \sin y\right] + g(x) \end{aligned}$$

where the $g(x)$ is an arbitrary function. The second C-R equation gives

$$\begin{aligned} v(x, y) &= -\int e^x \left[-y \sin y + (1 + x) \cos y\right] dx \\ &= e^x \left[y\ \sin y - x \cos y\right] + h(y) \end{aligned}$$

Notice that the integrations give the same result and the extra functions can be set to zero and we have

$$v(x, y) = e^x \left[y\ \sin y - x \cos y\right]$$

(b) Starting with a second order polynomial, $p_2(z) = (z - z_1)(z - z_2)$ where we will call the zeros of $p_2(z)$ z_1 and z_2. Now do the contour integral of $1/p_2(z)$ where C encloses the zeros. From the residue theorem, we have

$$\begin{aligned} \oint_C \frac{1}{p_2(z)}\, dz &= \oint_C \frac{1}{(z - z_1)(z - z_2)}\, dz \\ &= 2\pi i \left[\frac{1}{(z_1 - z_2)} + \frac{1}{(z_2 - z_1)}\right] \\ &= 0 \end{aligned}$$

Doing the same thing for a third order polynomial, we have

$$\begin{aligned} \oint_C \frac{1}{p_3(z)}\, dz &= \oint_C \frac{1}{(z - z_1)(z - z_2)(z - z_3)}\, dz \\ &= 2\pi i \left[\frac{1}{(z_1 - z_2)(z_1 - z_3)} + \frac{1}{(z_2 - z_1)(z_2 - z_3)} + \frac{1}{(z_3 - z_1)(z_3 - z_2)}\right] \\ &= 2\pi i \left[\frac{z_2 - z_3}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)} - \frac{z_1 - z_3}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)} + \frac{z_1 - z_2}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)}\right] \\ &= 0 \end{aligned}$$

You can also do this using the method of partial fractions, namely by writing

$$\frac{1}{(z-z_1)(z-z_2)} = \frac{1}{z_1 - z_2}\left[\frac{1}{z-z_1} - \frac{1}{z-z_2}\right]$$

and using the Cauchy integral formula for the (constant) function

$$f(z) = \frac{1}{z_1 - z_2}.$$

In particular, we have

$$\begin{aligned}\oint_C \frac{1}{p_2(z)}\,dz &= \oint_C \frac{f(z)}{z-z_1}\,dz - \oint_c \frac{f(z)}{z-z_2}\,dz \\ &= 2\pi i\left[f(z_1) - f(z_2)\right] \\ &= 0\end{aligned}$$

because $f(z)$ is a constant.

A similar thing can be shown for the third order polynomial.