The action for a gravitating, charged scalar field is

$$S = \int \sqrt{-\gamma}\, d^4x \left\{ \frac{R}{16\pi G} - (\mathcal{D}_\mu\phi)(\mathcal{D}^\mu\phi)^* - V(\phi,\phi^*) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right\}$$

where we have made the definitions

$$\begin{aligned} \mathcal{D}_\mu &= \nabla_\mu - ieA_\mu \\ F_{\mu\nu} &= \nabla_\mu A_\nu - \nabla_\nu A_\mu \\ \phi &= \phi_1 + i\phi_2 \end{aligned}$$

with the charge e being the coupling between the scalar field and the gauge field, A_μ.

The resulting equations of motion are

$$\begin{aligned} R_{\mu\nu} &= 8\pi G\left(T_{\mu\nu} - \frac{1}{2}\,\gamma_{\mu\nu}T_\lambda{}^\lambda\right) \\ \nabla^\mu F_{\mu\alpha} &= ie\left\{\phi^*\nabla_\alpha\phi - \phi\nabla_\alpha\phi^*\right\} + 2e^2\phi\phi^*A_\alpha \equiv j_\alpha \\ \nabla_\mu\nabla^\mu\phi &= 2ieA^\mu\nabla_\mu\phi + e^2\phi A_\mu A^\mu + ie\phi\nabla_\mu A^\mu + \frac{\partial V}{\partial\phi^*} \end{aligned}$$

where the stress tensor is

$$T_{\alpha\mu} = \big(\mathcal{D}_\alpha\phi\big)\big(\mathcal{D}_\mu\phi\big)^* + \big(\mathcal{D}_\alpha\phi\big)^*\big(\mathcal{D}_\mu\phi\big) - \gamma_{\alpha\mu}\left(\mathcal{D}_\lambda\phi\right)\big(\mathcal{D}^\lambda\phi\big)^* - \gamma_{\alpha\mu}V(\phi,\phi^*) + F_{\alpha\beta}F_\mu{}^\beta - \frac{1}{4}\,\gamma_{\alpha\mu}\,F_{\lambda\sigma}F^{\lambda\sigma}$$

We choose to work in the Lorentz gauge: $\nabla_\mu A^\mu = 0$.

The Maxwell equations in curved space can be written

$$\nabla^\mu\nabla_\mu A_\alpha = j_\alpha + 8\pi G\left(T_\alpha{}^\mu - \frac{1}{2}\,\gamma_\alpha{}^\mu\,T_\lambda{}^\lambda\right)A_\mu$$

where, in order to write the vector Laplacian of A_α, we have commuted covariant derivatives (which is how we picked up the Ricci tensor) and used the Lorentz gauge.

We want to perform the $(2+1)+1$ decomposition on these equations. In our original writeup, this was performed for the Einstein–scalar equations. Most of that can be taken over here, though the electromagnetic field will certainly modify those equations. Although we will try to write out all the relevant equations, much of our effort here will be focused on decomposing the electromagnetic field. We will rely heavily on our earlier definitions and the reader is referred to that for greater detail as to our approach and formalism.

The Maxwell equations

Recall that the metric on the 4-manifold can be written in terms of the other metrics

$$\begin{aligned} {}^{(4)}\gamma_{\mu\nu} &= {}^{(3)}g_{\mu\nu} + \frac{1}{s^2}X_\mu X_\nu \\ &= {}^{(2)}h_{\mu\nu} - n_\mu n_\nu + \frac{1}{s^2}X_\mu X_\nu \end{aligned}$$

where n^μ is the timelike unit vector normal to the spatial 2-hypersurface and X^μ is the Killing vector associated with the axisymmetry. The scalar field s defines the norm of the Killing vector: $X^\mu X_\mu = s^2$. Decomposing the covariant derivative of the gauge field, we get

$$\begin{aligned} \nabla_\mu A_\alpha &= \partial_\mu A_\alpha - {}^{(4)}\,\Gamma^\lambda_{\mu\alpha}A_\lambda \\ &= D_\mu A_\alpha - \frac{1}{2}s^2 g^{\sigma\lambda}\left[Y_\alpha Z_{\mu\sigma} + Y_\mu Z_{\alpha\sigma} - \partial_\sigma\big(\ln s^2\big)Y_\mu Y_\alpha\right]A_\lambda - \frac{1}{2}Y^\lambda A_\lambda\left[\partial_\mu\big(s^2Y_\alpha\big) + \partial_\alpha\big(s^2Y_\mu\big)\right] \end{aligned}$$

where we have defined $Y^\mu = X^\mu/s^2 = \delta^\mu_\varphi/s^2$ and $Z_{\mu\nu} = \partial_\mu Y_\nu - \partial_\nu Y_\mu$ with D_μ defined to be the covariant derivative on the 3-manifold with metric $g_{\mu\nu}$. With this we can calculate the vector Laplacian. What we

really want, however, is the parts of the vector Laplacian projected into the 3-manifold and along the Killing vector. After some calculation, these are

$$\begin{aligned} g_a{}^\alpha \cdot \nabla^\mu \nabla_\mu A_\alpha &= D^\mu D_\mu A_a + \frac{1}{s} D^\mu s D_\mu A_a - \frac{1}{s^2} D_\mu s A^\mu \, D_a s - D_\mu A_\varphi \, Z^\mu{}_a - \frac{1}{2s} D_\mu \big(s Z^\mu{}_a \big) \, A_\varphi \\ X^\alpha \cdot \nabla^\mu \nabla_\mu A_\alpha &= D^\mu D_\mu A_\varphi - \frac{1}{s} D^\mu \big(A_\varphi D_\mu s \big) - s^2 \, D^\mu A^\lambda \, Z_{\mu\lambda} - \frac{1}{2s} A^\lambda D_\mu \big(s^3 Z^\mu{}_\lambda \big) \\ &= D^\mu D_\mu A_\varphi - \frac{1}{s} D^\mu \big(A_\varphi D_\mu s \big) - s^2 \, D^\mu A^\lambda \, Z_{\mu\lambda} + 8\pi G \big(T^\lambda{}_\varphi - Y^\lambda T_{\varphi\varphi} \big) \, A_\lambda \end{aligned}$$

where, in the last line, we have used a result from our earlier writeup. At this point, we make what is a crucial observation for the consistency of our subsequent development. In our original considerations of axisymmetric gravitational collapse, our matter was only a scalar field (perhaps with a potential). However, this allowed a significant simplification at the time, namely the introduction of the scalar twist, or twist potential, w. This was defined by

$$\nabla_\mu w = w_\mu = \epsilon_{\mu\nu\lambda\sigma} X^\nu \nabla^\lambda X^\sigma$$

In the present case, this can be rewritten as

$$w_\mu = \frac{1}{2} s^4 \epsilon_{\mu\nu\lambda\sigma} Y^\nu Z^{\lambda\sigma}.$$

Our use of the scalar twist arose through one of the Einstein equations in the 3-manifold:

$$D^a (s^3 Z_{ba}) = 16\pi G s \, (T_{b\varphi} - Y_b T_{\varphi\varphi})$$

The twist, w_μ, can always be defined, but the scalar twist, w, can be defined provided the right hand side of the above Einstein equation is zero. For a nontrivial electromagnetic field, this quantity, $T_{b\varphi} - A_b T_{\varphi\varphi}$ will be zero if and only if $A_\varphi = 0$. If we make this assumption, it might be tempting to think that we can now use the scalar twist and all is well. However, note the above vector Laplacian projected along the Killing vector, X^α. This is an evolution equation for A_φ. Even when additional stress-energy terms are included, this equation is not always satisfied with $A_\varphi = 0$. Indeed, there is a term $s^2 Z_{\mu\lambda} D^\mu A^\lambda$ which will source A_φ unless it is identically zero as well. Because we want to allow A^λ to be nonzero, we must choose $Z_{\mu\lambda} = 0$. However, this defines our scalar twist and is nonzero if and only if there is rotation in the spacetime. Thus, to continue to use the scalar twist in our formalism together with an electromagnetic field forces us to consider only poloidal magnetic fields (*i.e.* $A_\varphi = 0$) with no rotation: $w = 0$. Certainly, we could change our approach to allow for rotating electromagnetic fields, but we would then have to use the twist, w_μ, and our formalism would be considerably modified. At this point, though it is certainly something of a sacrifice to temporarily give up on rotating electromagnetic fields, it seems to me that we have more than enough to do to consider just collapsing non-rotating fields.

Therefore, for consistency, we will set $A_\varphi = 0$ and $Z_{\mu\nu} = 0$ in the following. Note that we have thereby solved the $X^\alpha \cdot \nabla^\mu \nabla_\mu A_\alpha$ equation exactly.

We now do the ADM decomposition. Considering the vector Laplacian in the 3-manifold, we first project it in the timelike direction, *i.e.* along n^a,

$$\begin{aligned} n^a g_a{}^\alpha \cdot \nabla^\mu \nabla_\mu A_\alpha = \overbrace{D_\mu D^\mu \big(n^a A_a \big)}^{(1)} - 2 \overbrace{D_\mu n^a \, D^\mu A_a}^{(2)} - \overbrace{D_\mu D^\mu n^a \cdot A_a}^{(3)} \\ + \frac{1}{s} D^\mu s \, \big(\overbrace{D_\mu (n^a A_a)}^{(4)} - \overbrace{A_a \, D_\mu n^a}^{(5)} \big) - \overbrace{\frac{1}{s^2} A^\mu D_\mu s \cdot n^a D_a s}^{(6)} \end{aligned}$$

where, because we are now living fully in the 3-manifold, we can switch from using Greek indices $(\mu, \lambda, \cdots)$ to using Latin indices $(a, b, \cdots)$ which range over $0, 1$, and 2. Note the numbering above each term. In the

hope of keeping things somewhat orderly, let's consider each term in turn:

$$(1) \qquad D_b D^b\big(n^a A_a\big) = h^{ab}\Delta_a\Delta_b(nA) + K\cdot n^a D_a(nA) - n^a D_a\big[n^b D_b(nA)\big] + n^a D_a n^b \cdot D_b(nA)$$

$$(2) \qquad D_b n^a\, D^b A_a = -K^{ab} D_a A_b - n^b D_b n_a\, n^c D_c A^a$$

$$(3) \qquad D_b D^b n^a \cdot A_a = A^c \cdot g^{ab}\Delta_a\Delta_b n_c + (nA)K^{ab}K_{ab} - K_{ac}A^c \cdot n^b D_b n^a + K\cdot A^a n^b D_b n_a$$
$$- n^a D_a\big[n^b D_b(nA)\big] + n^a D_a\big(n^b n^c D_c A_b\big) + n^a D_a A^b \cdot n^c D_c n_b$$

$$(4) \qquad \frac{1}{s} D^b s \cdot D_b(nA) = \frac{1}{s}\, h^{ab}\Delta_a s\, \Delta_b(nA) + \chi\, n^a D_a(nA)$$

$$(5) \qquad \frac{1}{s} A_a\, D^b s\, D_b n^a = -\frac{1}{s}\, K^{ab} A_a\, D_b s + \chi\, n^a D_a n_b \cdot A^b$$

$$(6) \qquad \frac{1}{s^2} A^a D_a s\, n^b D_b s = -\frac{\chi}{s}\, h^{ab} A_a\, D_b s - \chi^2 \cdot (nA)$$

where we have used $s\chi = -n^a D_a s$ from our earlier definitions (recall that $\chi = {}^{(3)}K_\varphi{}^\varphi$ and $n_a = (-\alpha, 0, 0)$) and the 2-extrinsic curvature, $K_{ab} = -D_a n_b$ with trace $K = K_{ab}h^{ab}$.* (Recall, as well, that with maximal slicing, $K = -\chi$.) The derivative operator, Δ_a, is the covariant derivative on the 2 dimensional spatial hypersurfaces built out of the corresponding metric h_{ab}. We have also defined a new scalar variable (shorthand, really) $nA = n_a A^a$. With the exception of term #3, each of the above calculations is a relatively straightforward projection using $g_{ab} = h_{ab} - n_a n_b$.

All but two of the above terms are now expressed in terms of quantities on the 2-manifold. The exceptions are $n^a D_a n_b \cdot h^{bc}$ and $D_a A_b$. Defining $n_a = (-\alpha, 0, 0)$ such that $n^a = \frac{1}{\alpha}(1, \beta^1, \beta^2)$, these can be shown to be

$$n^a D_a n_b \cdot h^{bc} = h^{bc} \cdot \frac{\partial_b \alpha}{\alpha}$$
$$D_a A_b = \Delta_a A_b - n_a h_b{}^d \cdot n^c D_c A_d - n_b D_a(nA) - n_b A^d K_{ad} - n_a n_b n^c D_c n_d \cdot A^d$$

Putting all of this together, changing to partial derivatives where appropriate, and grouping according to terms involving A^μ and its derivatives, we get

$$\begin{aligned} n^a g_a{}^\alpha \cdot \nabla^\mu \nabla_\mu A_\alpha = & -n^a \partial_a\big(n^b n^c D_c A_b\big) \\ & + h^{ab}\Delta_a\Delta_b(nA) \\ & + \big(K + \chi\big)\big[n^a D_a\big(nA\big) - A^a \cdot n^b D_b n_a\big] \\ & + h^{ab}\,\partial_a(nA)\left[\frac{\partial_b\alpha}{\alpha} + \frac{\partial_b s}{s}\right] \\ & + (nA)\left[\chi^2 - K_{ab}K^{ab}\right] \\ & + 2K^{ab}\Delta_a A_b \\ & + h^{ab}\,\frac{\partial_b\alpha}{\alpha}\, n^c D_c A_a \\ & + A_a\left[-\Delta_b\Delta^b n^a + K^{ba}\left(\frac{\partial_b\alpha}{\alpha} + \frac{\partial_b s}{s}\right) + \chi\, h^{ba}\,\frac{\partial_b s}{s}\right] \end{aligned}$$

Of course, this is only one side of the Maxwell equation. The full equation is

$$n^a g_a{}^\alpha \cdot \nabla^\mu \nabla_\mu A_\alpha = n^a\left[j_a + 8\pi G\left(T_a{}^\mu - \frac{1}{2}\gamma_a{}^\mu\, T_\lambda{}^\lambda\right) A_\mu\right]$$

At this point, it is convenient to define some auxiliary variables. For evolution variables, we will use the spatial components of the gauge field: $h_a{}^b A_b$, (which, in our eventual cylindrical coordinate system, will be A_ρ and A_z). Their "conjugate variables" will be

$$\Pi_b \equiv -h_b{}^c n^a D_a A_c$$

* Note that this definition for the extrinsic curvature includes a minus sign which is different from the definition in Wald.

Note that this is a quantity in the spatial hypersurface and will only have two components. (Indeed, we could replace b with B where we use capital Latin letters to run over 1 and 2.) We will also take $nA \equiv n^a A_a$ as another evolution variable. It turns out to be a much more natural choice than, say, A_t. Its conjugate variable will be

$$\Pi_{(nA)} \equiv n^b n^a D_a A_b = n^a \partial_a \big(nA\big) - A^d \cdot n^c D_c n_d$$

It becomes clear that the equation we have been dealing with is the evolution equation for $\Pi_{(nA)}$. Finally, noting that the Laplacian of n^a in the above equation can be written

$$\begin{aligned} -\Delta_a \Delta^a n_b &= \Delta_a K^a{}_b \\ &= \Delta_b K - {}^{(3)}R_{ac} n^c h^a{}_b \\ &= \Delta_b \big(K + \chi\big) + h^a{}_b \, \chi \frac{\partial_a s}{s} - K_b{}^a \frac{\partial_a s}{s} - 8\pi G \, T_{ac} \cdot n^c h^a{}_b \end{aligned}$$

we can combine these facts and write out the evolution equation as

$$\begin{aligned} n^a \partial_a \Pi_{(nA)} \equiv n^a \partial_a \big(n^b n^c D_c A_b\big) = h^{ab} \Delta_a \Delta_b (nA) \\ + \big(K + \chi\big) \Pi_{(nA)} + h^{ab} A_a \, \Delta_b \big(K + \chi\big) \\ + h^{ab} \, \partial_a (nA) \, \frac{\partial_b \big(\alpha s\big)}{\alpha s} \\ + (nA) \left[\chi^2 - K_{ab} K^{ab}\right] \\ + 2 K^{ab} \Delta_a A_b \\ - h^{ab} \, \frac{\partial_a \alpha}{\alpha} \, \Pi_b \\ + A_a \left[K^{ba} \, \frac{\partial_b \alpha}{\alpha} + 2\chi \, h^{ba} \, \frac{\partial_b s}{s}\right] \\ - n^a j_a - 8\pi G \left[2 T_{ab} n^a h^{bc} A_c - (nA) \, n^a n^b T_{ab} - \frac{1}{2} \, (nA) \, T_\lambda{}^\lambda\right] \end{aligned}$$

We consider now, the same quantity: $g_a{}^\alpha \cdot \nabla^\mu \nabla_\mu A_\alpha$, but now projected into the spatial hypersurface,

$$h_b{}^a \cdot g_a{}^\alpha \cdot \nabla^\mu \nabla_\mu A_\alpha = \overbrace{h_b{}^a \, D_c D^c A_a}^{(7)} + \overbrace{h_b{}^a \, \frac{1}{s} D^c s D_c A_a}^{(8)} - \overbrace{h_b{}^a \, \frac{1}{s^2} D_a s D_c s A^c}^{(9)}$$

Again, working on the terms individually, we find

$$\begin{aligned} (7) \qquad h_b{}^a \, D_c D^c A_a = g^{cd} \Delta_c \Delta_d A_b - K \, \Pi_b + h^a_b \cdot n^c D_c \Pi_a \\ + n^a D_a n^b \big(\Delta_b A_c + h_{bc} \Pi_{(nA)}\big) + K_b{}^c \left[\partial_c \big(nA\big) + A_d K_c{}^d\right] \end{aligned}$$

$$(8) \qquad h_b{}^a \, \frac{1}{s} D^c s D_c A_a = h^{ac} \frac{\partial_a s}{s} \Delta_c A_b - \chi \, \Pi_b$$

$$(9) \qquad h_b{}^a \, \frac{1}{s^2} D_a s D_c s A^c = \frac{1}{s^2} \, h_b{}^a D_a s \, h^{cd} D_c s \, A_d + \chi \big(nA\big) \, \frac{1}{s} \, h_b{}^a D_a s$$

So that on combining these we get

$$\begin{aligned} h_b{}^a \cdot g_a{}^\alpha \cdot \nabla^\mu \nabla_\mu A_\alpha = h_b{}^a \cdot n^c D_c \Pi_a + h^{ac} \Delta_a \Delta_c A_b - \big(K + \chi\big) \Pi_b \\ + \Pi_{(nA)} \, h_b{}^a \frac{\partial_a \alpha}{\alpha} \\ + h^{ac} \Delta_c A_b \, \frac{\partial_a \big(\alpha s\big)}{\alpha s} \\ + K_b{}^c \left[\partial_c \big(nA\big) + K_c{}^d A_d\right] \\ - \frac{1}{s} \, h_b{}^a D_a s \left[\frac{1}{s} \, h^{cd} A_d D_c s + \chi \, \big(nA\big)\right] \end{aligned}$$

Equating with the appropriate right hand side,

$$\begin{aligned} h_b{}^a \cdot g_a{}^\alpha \cdot \nabla^\mu \nabla_\mu A_\alpha &= h_b{}^a \left[j_a + 8\pi G \left(T_a{}^\mu - \frac{1}{2} \gamma_a{}^\mu T_\lambda{}^\lambda \right) A_\mu \right] \\ &= h_b{}^a \, j_a + 8\pi G \left[h_b{}^a h^{cd} T_{ac} A_d - h_b{}^a n^c T_{ac} \big(nA\big) - \frac{1}{2} h_b{}^a A_a \, T_\lambda{}^\lambda \right] \end{aligned}$$

Because this is the evolution equation for Π_b, we can write

$$\begin{aligned} h_b{}^a \, n^d D_d \Pi_a = & -\Delta_a \Delta^a A_b + \big(K + \chi\big) \Pi_b \\ & - \Pi_{(nA)} \, h_b{}^a \frac{\partial_a \alpha}{\alpha} \\ & - h^{ac} \Delta_c A_b \, \frac{\partial_a \big(\alpha s\big)}{\alpha s} \\ & - K_b{}^c \left[\partial_c \big(nA\big) + K_c{}^d A_d \right] \\ & + \frac{1}{s} \, h_b{}^a D_a s \left[\frac{1}{s} \, h^{cd} A_d D_c s + \chi \left(nA \right) \right] \\ & + h_b{}^a \, j_a + 8\pi G \left[h_b{}^a h^{cd} T_{ac} A_d - h_b{}^a n^c T_{ac} \left(nA \right) - \frac{1}{2} h_b{}^a A_a \, T_\lambda{}^\lambda \right] \end{aligned}$$

With this, it might seem that we're finished with the Maxwell equations. Actually, we still need to consider two equations. These are the Lorentz gauge condition and the Gauss constraint, *i.e.* the curved space generalization of Gauss' Law. The first is fairly straightforward and comes right out of our decomposed covariant derivative

$$\begin{aligned} 0 &= \nabla_\mu A^\mu \\ &= \Delta_a A^a - n^a \partial_a \big(nA\big) + A^d n^c D_c n_d + h^{ab} A_a \frac{\partial_b s}{s} + \chi \big(nA\big) \\ &= \Delta_a A^a - \Pi_{(nA)} + h^{ab} A_a \frac{\partial_b s}{s} + \chi \big(nA\big) \end{aligned}$$

The Gauss constraint defines the initial value problem and must be solved on the first time slice in order to provide good initial data. It arises because the $\alpha = t$ component of the original Maxwell equation is not an evolution equation but an elliptic equation. Another way to think of it is that $\nabla^\mu F_{\mu t} = \nabla^\mu \nabla_\mu A_t - \nabla^\mu \nabla_t A_\mu$ has no second time derivatives of A_t (or any other component of A_μ).

To deal with this, we must go back to the original Maxwell equations and decompose

$$g_a{}^\alpha \big(\nabla^\mu \nabla_\mu A_\alpha - \nabla^\mu \nabla_\alpha A_\mu \big) = g_a{}^\alpha j_\alpha$$

along n^a. The first term we have already done. The second term takes some work, but we can show that the entire expression becomes

$$n^a \cdot g_a{}^b \big(D_c D^c A_b - D_c D_b A^c \big) + \frac{1}{s} \, n^a g_a{}^b \, D^c s \big(D_c A_b - D_b A_c \big) - n^a g_a{}^b \, D^c A_\varphi \cdot Z_{cb} = n^a g_a{}^b j_b$$

where, if we assume $A_\varphi = 0$ and $Z_{\mu\nu} = 0$, the third term drops out. Note too that the middle term in parentheses is just F_{cb}, the Maxwell tensor projected onto the 3–manifold. Rewriting the second derivatives we have

$$\begin{aligned} n^a D_b D^b A_a - n^a D_b D_a A^b = & \big(\overbrace{D_b D^b \big(n^a A_a \big)}^{(1)} - 2 \overbrace{D_b n^a \, D^b A_a}^{(2)} - \overbrace{D_b D^b n^a \cdot A_a}^{(3)} \big) \\ & - \big(\overbrace{D_b \big(n^a D_a A^b \big)}^{(10)} - \overbrace{D_b n^a \, D_a A^b}^{(11)} \big) \end{aligned}$$

where the first three terms were calculated earlier. The other two become

$$D_b\big(n^a D_a A^b\big) = -\Delta_b \Pi^b - \Pi_a \cdot n^b D_b n^a + K \cdot \Pi_{(nA)} - n^a D_a \Pi_{(nA)} \tag{10}$$

$$D_b n^a \, D_a A^b = -K^{ab} D_b A^a - h^{ab} \partial_a \big(nA\big) n^c D_c n_b - K^{ab} A_a \cdot n^c D_c n_b \tag{11}$$

Putting it all together and using our expression for $-\Delta_a \Delta^a n_b$ again, we get the following elliptic equation for nA:

$$\begin{aligned} \Delta_a \Delta^a \big(nA\big) = &-K^{ab} \Delta_a A_b - A^c \cdot \Delta_c \big(K + \chi\big) \\ &+ \big(nA\big) K^{ab} K_{ab} \\ &- h^{ab} \Delta_a \Pi_b \\ &+ \frac{\Delta_a s}{s} \big[A^b \big(K^a{}_b - h^a{}_b \chi\big) - F^a{}_b \, n^b\big] \\ &+ n^a j_a + 8\pi G \, n^a h^b{}_c T_{ab} A^c \end{aligned}$$

Collecting all the relevant equations from this section, we have the evolution equation for $\Pi_{(nA)}$ (the Maxwell equation projected along n^a)

$$\begin{aligned} n^a D_a \Pi_{(nA)} = &\, h^{ab} \Delta_a \Delta_b (nA) + \big(K + \chi\big) \Pi_{(nA)} + h^{ab} A_a \, \Delta_b \big(K + \chi\big) \\ &+ h^{ab} \, \partial_a (nA) \, \frac{\partial_b \big(\alpha s\big)}{\alpha s} + (nA) \left[\chi^2 - K_{ab} K^{ab}\right] + 2 K^{ab} \Delta_a A_b \\ &- h^{ab} \, \frac{\partial_a \alpha}{\alpha} \, \Pi_b + A_a \left[K^{ba} \, \frac{\partial_b \alpha}{\alpha} + 2\chi \, h^{ba} \, \frac{\partial_b s}{s}\right] \\ &- n^a j_a - 8\pi G \left[2 T_{ab} n^a h^{bc} A_c - (nA) \, n^a n^b T_{ab} - \frac{1}{2} \, (nA) \, T_\lambda{}^\lambda\right] \end{aligned}$$

the evolution equation for Π_b (the Maxwell equation projected into the spatial 2–hypersurface)

$$\begin{aligned} h_b{}^a \, n^d D_d \Pi_a = &-\Delta_a \Delta^a A_b + \big(K + \chi\big) \Pi_b - \Pi_{(nA)} \, h_b{}^a \frac{\partial_a \alpha}{\alpha} - h^{ac} \Delta_c A_b \, \frac{\partial_a \big(\alpha s\big)}{\alpha s} \\ &- K_b{}^c \left[\partial_c \big(nA\big) + K_c{}^d A_d\right] + \frac{1}{s} \, h_b{}^a D_a s \left[\frac{1}{s} \, h^{cd} A_d D_c s + \chi \, \big(nA\big)\right] \\ &+ h_b{}^a \, j_a + 8\pi G \left[h_b{}^a h^{cd} T_{ac} A_d - h_b{}^a n^c T_{ac} \, \big(nA\big) - \frac{1}{2} h_b{}^a A_a \, T_\lambda{}^\lambda\right] \end{aligned}$$

the Lorentz gauge (constraint) condition

$$0 = \Delta_a A^a - \Pi_{(nA)} + h^{ab} A_a \frac{\partial_b s}{s} + \chi\big(nA\big)$$

and the Gauss constraint

$$\begin{aligned} \Delta_a \Delta^a \big(nA\big) = &-K^{ab} \Delta_a A_b - A^c \cdot \Delta_c \big(K + \chi\big) + \big(nA\big) K^{ab} K_{ab} - h^{ab} \Delta_a \Pi_b \\ &+ \frac{\Delta_a s}{s} \big[A^b \big(K^a{}_b - h^a{}_b \chi\big) - F^a{}_b \, n^b\big] + n^a j_a + 8\pi G \, n^a h^b{}_c T_{ab} A^c \end{aligned}$$

The scalar field equation

The equation for the scalar field is also changed with the addition of electromagnetism. So let's revisit it. The general equation is

$$\nabla_\mu \nabla^\mu \phi = 2ieA^\mu \nabla_\mu \phi + e^2 \phi A_\mu A^\mu + ie\phi \nabla_\mu A^\mu + \frac{\partial V}{\partial \phi^*}$$

where we will *not* drop the term with $\nabla_\mu A^\mu$ because we will use it in our definition of Π_ϕ. First rewrite in a slightly more convenient form

$$\nabla^\mu\big(\nabla_\mu\phi - ieA_\mu\phi\big) = ieA^\mu\big(\nabla_\mu\phi - ieA_\mu\phi\big) + \frac{\partial V}{\partial\phi^*}$$

Decomposing this equation, we get

$$\begin{aligned} n^c D_c\Pi_\phi = \Delta_a\left(\Delta^a\phi - ieA^a\phi\right) + ie\big(nA\big)\left[\Pi_\phi + K\phi\right] \\ + h^{ab}\left[\frac{\partial_a(\alpha s)}{\alpha s} - ieA_a\right]\left(\partial_b\phi - ieA_b\phi\right) - \frac{\partial V}{\partial\phi^*} \end{aligned}$$

where we have defined

$$\Pi_\phi = n^d\big(\partial_d\phi - ieA_d\phi\big)$$

to be the variable conjugate to ϕ.

The stress tensor

Now consider the stress tensor. It is, again,

$$T_{\alpha\mu} = \big(\mathcal{D}_\alpha\phi\big)\big(\mathcal{D}_\mu\phi\big)^* + \text{c.c.} - \gamma_{\alpha\mu}\left(\mathcal{D}_\lambda\phi\right)\big(\mathcal{D}^\lambda\phi\big)^* - \gamma_{\alpha\mu}V(\phi,\phi^*) + F_{\alpha\beta}F_\mu{}^\beta - \frac{1}{4}\,\gamma_{\alpha\mu}\,F_{\lambda\sigma}F^{\lambda\sigma}$$

where c.c. denotes the complex conjugate of the preceding term. Decomposing the Maxwell tensor yields

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= \big(h_\mu{}^\lambda - n_\mu n^\lambda + \frac{1}{s^2}X_\mu X^\lambda\big)\big(h_\nu{}^\sigma - n_\nu n^\sigma + \frac{1}{s^2}X_\nu X^\sigma\big)\big(\partial_\lambda A_\sigma - \partial_\sigma A_\lambda\big) \\ &= \mathcal{B}_{\mu\nu} + 2n_{[\mu}h_{\nu]}{}^\lambda\,\mathcal{E}_\lambda - \frac{2}{s^2}X_{[\mu}\partial_{\nu]}A_\varphi \end{aligned}$$

where we define the auxiliary quantities

$$\begin{aligned} \mathcal{B}_{\mu\nu} &= h_\mu{}^\lambda h_\nu{}^\sigma\big(\partial_\lambda A_\sigma - \partial_\sigma A_\lambda\big) \\ \mathcal{E}_\lambda &= D_\lambda\big(nA\big) + K_{\lambda\mu}A^\mu + \Pi_\lambda + n_\lambda\cdot n^\sigma D_\sigma\big(nA\big) \end{aligned}$$

Note that both $\mathcal{B}_{\mu\nu}$ and $\mathcal{E}_\mu$ live in the spatial hypersurfaces ($n^\mu\mathcal{E}_\mu = n^\mu\mathcal{B}_{\mu\nu} = 0$). In addition, we have $\mathcal{B}_{\mu\nu} = -\mathcal{B}_{\nu\mu}$. Thus $\mathcal{B}_{\mu\nu}$ has only one independent component. We can now write

$$\begin{aligned} F_{\alpha\beta}F_\mu{}^\beta &= h^{\beta\nu}\left[\mathcal{B}_{\alpha\beta} + n_\alpha h_\beta{}^{\bar\lambda}\mathcal{E}_{\bar\lambda} - \frac{1}{s^2}X_\alpha\partial_\beta A_\varphi\right]\left[\mathcal{B}_{\mu\nu} + n_\mu h_\nu{}^\lambda\mathcal{E}_\lambda - \frac{1}{s^2}X_\mu\partial_\nu A_\varphi\right] \\ &\quad - \left[h_\alpha{}^{\bar\lambda}\mathcal{E}_{\bar\lambda} - \frac{1}{s^2}X_\alpha n^\beta\partial_\beta A_\varphi\right]\left[h_\mu{}^\lambda\mathcal{E}_\lambda - \frac{1}{s^2}X_\mu n^\nu\partial_\nu A_\varphi\right] \\ &\quad + \frac{1}{s^2}\partial_\alpha A_\varphi\partial_\mu A_\varphi \\ F_{\alpha\beta}F^{\alpha\beta} &= h^{\beta\nu}\mathcal{B}_{\mu\beta}\mathcal{B}^\mu{}_\nu - 2h^{\beta\nu}h_\beta{}^{\bar\lambda}h_\nu{}^\lambda\mathcal{E}_{\bar\lambda}\mathcal{E}_\lambda \\ &\quad + \frac{2}{s^2}h^{\beta\nu}\partial_\beta A_\varphi\partial_\nu A_\varphi - \frac{2}{s^2}\,n^\beta\partial_\beta A_\varphi\cdot n^\nu\partial_\nu A_\varphi \end{aligned}$$

Since our use of the scalar twist requires that $A_\varphi = 0$, we'll drop those terms in the stress tensor leaving

$$\begin{aligned} T_{\alpha\mu} &= \big(\mathcal{D}_\alpha\phi\big)\big(\mathcal{D}_\mu\phi\big)^* + \text{c.c.} - \gamma_{\alpha\mu}\left[\big(\mathcal{D}_\lambda\phi\big)\big(\mathcal{D}^\lambda\phi\big)^* + V(\phi,\phi^*)\right] \\ &\quad + h^{\beta\nu}\big(\mathcal{B}_{\alpha\beta} + n_\alpha\mathcal{E}_\beta\big)\left(\mathcal{B}_{\mu\nu} + n_\mu\mathcal{E}_\nu\right) - \mathcal{E}_\alpha\mathcal{E}_\mu - \frac{1}{4}\,\gamma_{\alpha\mu}\big(\mathcal{B}_{\beta\nu}\mathcal{B}^{\beta\nu} - 2\,\mathcal{E}_\lambda\mathcal{E}^\lambda\big) \end{aligned}$$

Considering the matter terms in the previous Maxwell equations and with an eye to the source terms in the Einstein equations, some useful quantities to have are

$$\begin{aligned} T_\lambda{}^\lambda &= \gamma^{\alpha\mu}\,T_{\alpha\mu} \\ T_{\varphi\varphi} &= X^\alpha X^\mu\,T_{\alpha\mu} \\ \rho_{\text{mat}} &= n^\alpha n^\mu T_{\alpha\mu} = n^a n^b T_{ab} \\ -J_c &= n^a h^b{}_c\,T_{ab} \\ S_{bd} &= h^a{}_b\,h^c{}_d\,T_{ac} \end{aligned}$$

we can write

$$\begin{aligned} T_\lambda{}^\lambda &= 2\big[\,\big|n^a\mathcal{D}_a\phi\,\big|^2 - h^{ab}\mathcal{D}_a\phi\big(\mathcal{D}_b\phi\big)^*\,\big] - 4V(\phi,\phi^*) \\ T_{\varphi\varphi} = X^\alpha X^\mu\,T_{\alpha\mu} &= -s^2\left[h^{ac}\mathcal{D}_a\phi\big(\mathcal{D}_c\phi\big)^* - \big|n^a\mathcal{D}_a\phi\,\big|^2 + V(\phi,\phi^*)\right] - \frac{s^2}{4}\left[\mathcal{B}_{ac}\mathcal{B}^{ac} - 2\,\mathcal{E}_a\mathcal{E}^a\right] \\ \rho_{\text{mat}} = n^a n^b T_{ab} &= \big|n^a\mathcal{D}_a\phi\,\big|^2 + h^{ab}\,\mathcal{D}_a\phi\big(\mathcal{D}_b\phi\big)^* + V(\phi,\phi^*) + \frac{1}{4}\left(\mathcal{B}_{ab}\mathcal{B}^{ab} + 2\,\mathcal{E}_a\mathcal{E}^a\right) \\ -J_c = n^a h^b{}_c\,T_{ab} &= h^b{}_c\mathcal{D}_b\phi\big(n^a\mathcal{D}_a\phi\big)^* + \text{c.c.} - h^b{}_c\mathcal{B}_{ba}\,\mathcal{E}^a \\ S_{bd} = h^a{}_b\,h^c{}_d\,T_{ac} &= h^a{}_b\,h^c{}_d\big[\mathcal{D}_a\phi\big(\mathcal{D}_c\phi\big)^* + \text{c.c.}\big] - h_{bd}\left\{h^{ac}\mathcal{D}_a\phi\big(\mathcal{D}_c\phi\big)^* - \big|n^a\mathcal{D}_a\phi\,\big|^2 + V(\phi,\phi^*)\right\} \\ &\quad + h^{ac}\mathcal{B}_{ba}\mathcal{B}_{dc} - \mathcal{E}_b\mathcal{E}_d - \frac{1}{4}\,h_{bd}\left[\mathcal{B}_{ac}\mathcal{B}^{ac} - 2\,\mathcal{E}_a\mathcal{E}^a\right] \end{aligned}$$

At this point, we could go back to our equations, add these matter terms and write out the equations in their scalar, to-be-differenced form. However, there are two potentially "ugly" terms. This, of course, is a matter of preference, but if we want to keep the number of terms in our equations to a minimum (admittedly **not** a particularly compelling reason if one's primary objective is a stable code), the term

$$\Delta^a \Delta_a A_b$$

in the evolution equation for Π_a and the term

$$2K^{ab}\,\Delta_a A_b$$

in the evolution equation for $\Pi_{(nA)}$ both produce a bevy of terms when written out in our coordinate system. If we accept this minimalist notion, the first term can be dealt with by introducing the commutator of two 2–covariant derivatives and the corresponding Ricci tensor

$$\Delta^a \Delta_a A_b = \Delta^a \big(\Delta_a A_b - \Delta_b A_a\big) + {}^{(2)}R^a{}_b\, A_a + \partial_b\big(\Delta^a A_a\big)$$

and noticing that the first term in parentheses is nothing but the 2–covariant derivative of one of our auxiliary variables: $\Delta^a \mathcal{B}_{ab}$. This is progress since the 2–Ricci tensor is very simple in our chosen coordinate system.

Using this in our evolution equation for Π_b together with the Lorentz condition and the definition of $\mathcal{E}_a$ to eliminate the $K_b{}^c K_c{}^d$ term, we have

$$\begin{aligned} h_b{}^a\, n^d D_d \Pi_a = &-\Delta^a \mathcal{B}_{ab} - {}^{(2)}R^a{}_b\, A_a - \partial_b\big(\Delta^a A_a\big) + \big(K+\chi\big)\,\Pi_b \\ &- \Pi_{(nA)}\, h_b{}^a \frac{\partial_a \alpha}{\alpha} - h^{ac}\Delta_c A_b\, \frac{\partial_a(\alpha s)}{\alpha s} \\ &- K_b{}^c\big(\mathcal{E}_c - \Pi_c\big) + \frac{\partial_b s}{s}\left[\Pi_{(nA)} - \Delta_a A^a\right] \\ &+ h_b{}^a\, j_a + 8\pi G\left[h_b{}^a h^{cd} T_{ac}\, A_d - h_b{}^a n^c T_{ac}\,(nA) - \frac{1}{2}\, h_b{}^a A_a\, T_\lambda{}^\lambda\right] \end{aligned}$$

For the second "ugly" term, we note that a term like it is also present in the Gauss constraint. If we thus use the Gauss constraint and the definition of $\mathcal{E}_a$, we can write the evolution equation for $\Pi_{(nA)}$ as

$$\begin{aligned} n^a D_a \Pi_{(nA)} = &-\Delta_a \Delta^a (nA) + \big(K+\chi\big)\,\Pi_{(nA)} - h^{ab} A_a\, \Delta_b\big(K+\chi\big) - 2\,\Delta_a \Pi^a \\ &- h^{ab}\,\partial_a (nA)\,\frac{\partial_b s}{s} + (nA)\left[\chi^2 + K_{ab}K^{ab}\right] - 2\,\Pi^a\,\frac{\partial_a(\alpha s)}{\alpha s} \\ &+ \mathcal{E}_b\, n^a D_a n^b + n^a j_a + 8\pi G\,(nA)\left[n^a n^b T_{ab} + \frac{1}{2}\,T_\lambda{}^\lambda\right] \end{aligned}$$

Making a final observation, we can express the Gauss constraint in terms of derivatives of $\mathcal{E}_a$. Using the definition

$$\mathcal{E}_a = D_a\big(nA\big) + K_{ab}A^b + \Pi_a + n_a \cdot n^b D_b\big(nA\big)$$

taking a 2–covariant derivative and using the Gauss constraint, we can write the Gauss constraint as

$$\frac{1}{s}\,\Delta_a\big(s\mathcal{E}^a\big) = n^a j^a.$$

This is nice since it accords with what one might expect from the curved space Maxwell equations so it is a nice check of our method.

All the matter equations in 2-covariant form (general stress–energy)

Collecting all of the matter equations together again and using generic terms from the stress-energy, we have the following evolution equations

$$
\begin{aligned}
n^a \partial_a \big(nA\big) &= A_a \, h^{ab} \frac{\partial_b \alpha}{\alpha} + \Pi_{(nA)} \\
n^a D_a \Pi_{(nA)} &= -\Delta_a \Delta^a (nA) + \big(K + \chi\big) \, \Pi_{(nA)} - h^{ab} A_a \, \Delta_b \big(K + \chi\big) - 2 \, \Delta_a \Pi^a \\
&\quad - h^{ab} \, \partial_a (nA) \, \frac{\partial_b s}{s} + (nA) \left[\chi^2 + K_{ab} K^{ab}\right] - 2 \, \Pi^a \, \frac{\partial_a \big(\alpha s\big)}{\alpha s} \\
&\quad + \mathcal{E}_b \, n^a D_a n^b + n^a j_a + 8\pi G \, \big(nA\big) \left[n^a n^b T_{ab} + \frac{1}{2} \, T_\lambda{}^\lambda \right] \\
h_b{}^c n^a D_a A_c &= -\Pi_b \\
h_b{}^a \, n^d D_d \Pi_a &= -\Delta^a \mathcal{B}_{ab} - {}^{(2)}R^a{}_b \, A_a - \partial_b \big(\Delta^a A_a\big) + \big(K + \chi\big) \, \Pi_b \\
&\quad - \Pi_{(nA)} \, h_b{}^a \frac{\partial_a \alpha}{\alpha} - h^{ac} \Delta_c A_b \, \frac{\partial_a \big(\alpha s\big)}{\alpha s} \\
&\quad - K_b{}^c \big(\mathcal{E}_c - \Pi_c\big) + \frac{\partial_b s}{s} \left[\Pi_{(nA)} - \Delta_a A^a\right] \\
&\quad + h_b{}^a \, j_a + 8\pi G \left[h_b{}^a h^{cd} T_{ac} \, A_d - h_b{}^a n^c T_{ac} \, \big(nA\big) - \frac{1}{2} \, h_b{}^a A_a \, T_\lambda{}^\lambda \right] \\
n^a \partial_a \phi &= ie \big(nA\big) \phi + \Pi_\phi \\
n^a \partial_a \Pi_\phi &= \Delta_a \left(\Delta^a \phi - ie A^a \phi\right) + ie \big(nA\big) \left[\Pi_\phi + K \phi\right] \\
&\quad + h^{ab} \left[\frac{\partial_a (\alpha s)}{\alpha s} - ie A_a \right] \left(\partial_b \phi - ie A_b \phi\right) - \frac{\partial V}{\partial \phi^*}
\end{aligned}
$$

and constraint equations

$$
\begin{aligned}
0 &= \Delta_a A^a - \Pi_{(nA)} + h^{ab} A_a \frac{\partial_b s}{s} + \chi \big(nA\big) \\
\Delta_a \Delta^a \big(nA\big) &= -K^{ab} \Delta_a A_b - A^c \cdot \Delta_c \big(K + \chi\big) + \big(nA\big) K^{ab} K_{ab} - h^{ab} \Delta_a \Pi_b \\
&\quad + \frac{\partial_a s}{s} \big[A^b \big(K^a{}_b - h^a{}_b \chi \big) - F^a{}_b \, n^b \big] + n^a j_a + 8\pi G \, n^a h^b{}_c T_{ab} A^c
\end{aligned}
$$

(Note that we could use $\Delta_a (s \mathcal{E}^a) = s \, n^a j_a$ as the Gauss constraint instead of the last equation above.)

All the matter equations in 2-covariant form (with charged scalar field as stress–energy)

Again, we write all of the matter equations down but this time with our particular choice for the stress–energy tensor, namely that for a charged scalar field. The evolution equations are

$$
\begin{aligned}
n^a\partial_a\big(nA\big) &= A_a\, h^{ab}\frac{\partial_b\alpha}{\alpha} + \Pi_{(nA)} \\
n^a D_a\Pi_{(nA)} &= -\Delta_a\Delta^a(nA) + \big(K+\chi\big)\,\Pi_{(nA)} - h^{ab}A_a\,\Delta_b\big(K+\chi\big) - 2\,\Delta_a\Pi^a \\
&\quad - h^{ab}\,\partial_a(nA)\,\frac{\partial_b s}{s} + (nA)\left[\chi^2 + K_{ab}K^{ab}\right] - 2\,\Pi^a\,\frac{\partial_a\big(\alpha s\big)}{\alpha s} \\
&\quad + \mathcal{E}_b\, n^a D_a n^b + ie\big[\phi^*\Pi_\phi - \phi\Pi^*_\phi\big] \\
&\quad + 8\pi G\,\big(nA\big)\left[2\big|\Pi_\phi\big|^2 - V + \frac{1}{4}\left(\mathcal{B}_{ab}\mathcal{B}^{ab} + 2\mathcal{E}_a\mathcal{E}^a\right)\right] \\
h_b{}^c n^a D_a A_c &= -\Pi_b \\
h_b{}^a\, n^d D_d\Pi_a &= -\Delta^a\mathcal{B}_{ab} - {}^{(2)}R^a{}_b\,A_a - \partial_b\big(\Delta^a A_a\big) + \big(K+\chi\big)\,\Pi_b \\
&\quad - \Pi_{(nA)}\,h_b{}^a\frac{\partial_a\alpha}{\alpha} - h^{ac}\Delta_c A_b\,\frac{\partial_a\big(\alpha s\big)}{\alpha s} \\
&\quad - K_b{}^c\big(\mathcal{E}_c - \Pi_c\big) + \frac{\partial_b s}{s}\left[\Pi_{(nA)} - \Delta_a A^a\right] \\
&\quad + ieh_b{}^a\left[\phi^*\big(\phi_{,a}\big) - \phi\big(\phi^*_{,a}\big) - 2ieA_a\big|\phi\big|^2\right] \\
&\quad + 8\pi G\,h_b{}^a\Big\{h^{cd}\,A_d\big[\mathcal{D}_a\phi\,(\mathcal{D}_c\phi)^* + \text{c.c.}\big] + A_a\,V - \big(nA\big)\left[\mathcal{D}_a\phi\cdot\Pi^*_\phi + \text{c.c.}\right] \\
&\qquad\qquad + \mathcal{B}_{ac}\mathcal{B}^{dc}A_d - \mathcal{E}_a\mathcal{E}^c A_c - \frac{1}{4}\,A_a\big(\mathcal{B}_{cd}\mathcal{B}^{cd} - 2\mathcal{E}_c\mathcal{E}^c\big) + \big(nA\big)\mathcal{B}_{ac}\mathcal{E}^c\Big\} \\
n^a\partial_a\phi &= ie\big(nA\big)\phi + \Pi_\phi \\
n^a\partial_a\Pi_\phi &= \Delta_a\Delta^a\phi - ie\Delta_a\left(A^a\phi\right) + ie\big(nA\big)\left[\Pi_\phi + K\phi\right] \\
&\quad + h^{ab}\left[\frac{\partial_a(\alpha s)}{\alpha s} - ieA_a\right]\left(\partial_b\phi - ieA_b\phi\right) - \frac{\partial V}{\partial\phi^*}
\end{aligned}
$$

and the constraint equations are

$$
\begin{aligned}
0 &= \Delta_a A^a - \Pi_{(nA)} + h^{ab}A_a\frac{\partial_b s}{s} + \chi\big(nA\big) \\
\Delta_a\Delta^a\big(nA\big) &= -K^{ab}\Delta_a A_b - A^c\cdot\Delta_c\big(K+\chi\big) + \big(nA\big)K^{ab}K_{ab} - h^{ab}\Delta_a\Pi_b \\
&\quad + \frac{\partial_a s}{s}\left[A^b\big(K^a{}_b - h^a{}_b\chi\big) - \mathcal{E}^a\right] + ie\big[\phi^*\Pi_\phi - \phi\Pi^*_\phi\big] \\
&\quad + 8\pi G\,h^b{}_c A^c\big[\mathcal{D}_b\,\Pi^*_\phi + \text{c.c.} - \mathcal{B}_{ba}\mathcal{E}^a\big]
\end{aligned}
$$

(Where, again, we could use $\Delta_a(s\mathcal{E}^a) = s\,n^a j_a$ as the Gauss constraint instead of the last equation above.)

To these equations we must also add the Einstein equations with a charged scalar field as the matter source.

All the (unregularized) equations

Writing out all the equations in our assumed coordinates with maximal slicing, $(K+\chi=0)$, we have

$$
\begin{aligned}
\big(nA\big)_{,t} &= \beta^\rho \big(nA\big)_{,\rho} + \beta^z \big(nA\big)_{,z} + \alpha\,\Pi_{(nA)} + \frac{1}{a^2}\big[\alpha_{,\rho}A_\rho + \alpha_{,z}A_z\big] \\
\Pi_{(nA),t} &= \beta^\rho \Pi_{(nA),\rho} + \beta^z \Pi_{(nA),z} - \frac{\alpha}{a^2}\big[\big(nA\big)_{,\rho\rho} + \big(nA\big)_{,zz}\big] + \frac{2}{sa^2}\big[\big(\alpha s \Pi_\rho\big)_{,\rho} + \big(\alpha s \Pi_z\big)_{,z}\big] \\
&\quad + 2\alpha\big(nA\big)\big[{K_\rho}^{\rho 2} + {K_z}^{z 2} + {K_\rho}^{z 2} + {K_\rho}^\rho {K_z}^z\big] - \frac{\alpha}{a^2}\big[\big(nA\big)_{,\rho}\frac{s_{,\rho}}{s} + \big(nA\big)_{,z}\frac{s_{,z}}{s} - \mathcal{E}_\rho\frac{\alpha_{,\rho}}{\alpha} - \mathcal{E}_z\frac{\alpha_{,z}}{\alpha}\big] \\
&\quad - ie\alpha\big(\phi^*\Pi_\phi - \phi{\Pi_\phi}^*\big) + 8\pi G\,\alpha\big(nA\big)\left[2\,\big|\Pi_\phi\,\big|^2 - V + \frac{1}{2a^2}\big({\mathcal{E}_\rho}^2 + {\mathcal{E}_z}^2 + \frac{1}{a^2}{\mathcal{B}_{\rho z}}^2\big)\right] \\
A_{\rho,t} &= \beta^\rho A_{\rho,\rho} + \beta^z A_{\rho,z} - \big(nA\big)\alpha_{,\rho} - \alpha\Pi_\rho + A_\rho\,\big(-\alpha {K_\rho}^\rho + {\beta^\rho}_{,\rho}\big) + A_z\,\frac{1}{2}\big({\beta^z}_{,\rho} - {\beta^\rho}_{,z}\big) \\
\Pi_{\rho,t} &= \beta^\rho \Pi_{\rho,\rho} + \beta^z \Pi_{\rho,z} + \Pi_\rho\big(-\alpha {K_\rho}^\rho + {\beta^\rho}_{,\rho}\big) + \Pi_z\frac{1}{2}\big({\beta^z}_{,\rho} - {\beta^\rho}_{,z}\big) \\
&\quad + \alpha\big(\frac{1}{a^2}\mathcal{B}_{\rho z}\big)_{,z} - \frac{\alpha}{s}\big[\frac{s}{a^2}\big(A_{\rho,\rho} + A_{z,z}\big)\big]_{,\rho} - \frac{1}{sa^2}\big[\big(\alpha s\big)_{,\rho}A_{\rho,\rho} + \big(\alpha s\big)_{,z}A_{\rho,z}\big] \\
&\quad + A_\rho\frac{1}{sa^2}\left[\big(\frac{a_{,\rho}}{a}\,\alpha s\big)_{,\rho} + \big(\frac{a_{,z}}{a}\,\alpha s\big)_{,z}\right] - A_z\frac{1}{sa^3}\left[\big(\alpha s\big)_{,\rho}a_{,z} - \big(\alpha s\big)_{,z}a_{,\rho}\right] \\
&\quad + \alpha\Pi_{(nA)}\big(\frac{s_{,\rho}}{s} - \frac{\alpha_{,\rho}}{\alpha}\big) + \alpha {K_\rho}^\rho\big(\Pi_\rho - \mathcal{E}_\rho\big) + \alpha {K_\rho}^z\big(\Pi_z - \mathcal{E}_z\big) \\
&\quad + ie\alpha\big[\phi^*\mathcal{D}_\rho\phi - \phi\big(\mathcal{D}_\rho\phi\big)^*\big] \\
&\quad + 8\pi G\alpha\left\{\frac{2A_\rho}{a^2}\big|\mathcal{D}_\rho\phi\big|^2 + \frac{1}{a^2}\,A_z\,\big[\mathcal{D}_\rho\phi\big(\mathcal{D}_z\phi\big)^* + \text{c.c.}\big] + A_\rho\,V - \big(nA\big)\big[\mathcal{D}_\rho\phi\,\Pi_\phi^* + \text{c.c.}\big]\right. \\
&\qquad\qquad \left. + \frac{A_\rho}{2a^2}\big(\frac{{\mathcal{B}_{\rho z}}^2}{a^2} - {\mathcal{E}_\rho}^2 + {\mathcal{E}_z}^2\big) - \frac{A_z}{a^2}\mathcal{E}_\rho\mathcal{E}_z + \frac{\big(nA\big)}{a^2}\mathcal{E}_z\mathcal{B}_{\rho z}\right\} \\
A_{z,t} &= \beta^\rho A_{z,\rho} + \beta^z A_{z,z} - \big(nA\big)\alpha_{,z} - \alpha\Pi_z + A_\rho\,\frac{1}{2}\big({\beta^\rho}_{,z} - {\beta^z}_{,\rho}\big) + A_z\,\big(-\alpha {K_\rho}^\rho + {\beta^\rho}_{,\rho}\big) \\
\Pi_{z,t} &= \beta^\rho \Pi_{z,\rho} + \beta^z \Pi_{z,z} + \Pi_\rho\frac{1}{2}\big({\beta^\rho}_{,z} - {\beta^z}_{,\rho}\big) + \Pi_z\big(-\alpha {K_\rho}^\rho + {\beta^\rho}_{,\rho}\big) \\
&\quad - \alpha\big(\frac{1}{a^2}\mathcal{B}_{\rho z}\big)_{,\rho} - \frac{\alpha}{s}\big[\frac{s}{a^2}\big(A_{\rho,\rho} + A_{z,z}\big)\big]_{,z} - \frac{1}{sa^2}\big[\big(\alpha s\big)_{,\rho}A_{z,\rho} + \big(\alpha s\big)_{,z}A_{z,z}\big] \\
&\quad + A_z\frac{1}{sa^2}\left[\big(\frac{a_{,\rho}}{a}\,\alpha s\big)_{,\rho} + \big(\frac{a_{,z}}{a}\,\alpha s\big)_{,z}\right] + A_\rho\frac{1}{sa^3}\left[\big(\alpha s\big)_{,\rho}a_{,z} - \big(\alpha s\big)_{,z}a_{,\rho}\right] \\
&\quad + \alpha\Pi_{(nA)}\big(\frac{s_{,z}}{s} - \frac{\alpha_{,z}}{\alpha}\big) + \alpha {K_z}^\rho\big(\Pi_\rho - \mathcal{E}_\rho\big) + \alpha {K_z}^z\big(\Pi_z - \mathcal{E}_z\big) \\
&\quad + ie\alpha\big[\phi^*\mathcal{D}_z\phi - \phi\big(\mathcal{D}_z\phi\big)^*\big] \\
&\quad + 8\pi G\alpha\left\{\frac{2A_z}{a^2}\big|\mathcal{D}_z\phi\big|^2 + \frac{1}{a^2}\,A_\rho\,\big[\mathcal{D}_\rho\phi\big(\mathcal{D}_z\phi\big)^* + \text{c.c.}\big] + A_z\,V - \big(nA\big)\big[\mathcal{D}_z\phi\,\Pi_\phi^* + \text{c.c.}\big]\right. \\
&\qquad\qquad \left. + \frac{A_z}{2a^2}\big(\frac{{\mathcal{B}_{\rho z}}^2}{a^2} + {\mathcal{E}_\rho}^2 - {\mathcal{E}_z}^2\big) - \frac{A_\rho}{a^2}\mathcal{E}_\rho\mathcal{E}_z - \frac{\big(nA\big)}{a^2}\mathcal{E}_\rho\mathcal{B}_{\rho z}\right\} \\
\phi_{,t} &= \beta^\rho\phi_{,\rho} + \beta^z\phi_{,z} + ie\alpha\big(nA\big)\phi + \alpha\Pi_\phi \\
\Pi_{\phi,t} &= \beta^\rho \Pi_{\phi,\rho} + \beta^z \Pi_{\phi,z} + ie\alpha\big(nA\big)\,\big(\Pi_\phi - \phi\chi\big) - \alpha\frac{\partial V}{\partial\phi^*} \\
&\quad + \frac{\alpha}{a^2}\big(\phi_{,\rho\rho} + \phi_{,zz}\big) - ie\frac{\alpha}{a^2}\big[\big(\phi A_\rho\big)_{,\rho} + \big(\phi A_z\big)_{,z}\big] \\
&\quad + \frac{\alpha}{a^2}\left[\left(\frac{(\alpha s)_{,\rho}}{\alpha s} - ieA_\rho\phi\right)\big(\phi_{,\rho} - ieA_\rho\phi\big) + \left(\frac{(\alpha s)_{,z}}{\alpha s} - ieA_z\phi\right)\big(\phi_{,z} - ieA_z\phi\big)\right]
\end{aligned}
$$

The constraint equations become

$$0 = \frac{1}{a^2}\big(A_{\rho,\rho} + A_{z,z}\big) - \Pi_{(nA)} + \frac{1}{sa^2}\big(A_\rho s_{,\rho} + A_z s_{,z}\big) + \big(nA\big)\chi$$

In addition we have a "Gauss constraint"

$$\begin{aligned}
0 = {}& \big(nA\big)_{,\rho\rho} + \big(nA\big)_{,zz} \\
&+ \Pi_{\rho,\rho} + \Pi_{z,z} - a^2\chi\Pi_{(nA)} \\
&- a^2\big(nA\big)\big[{K_\rho}^{\rho 2} + {K_z}^{z 2} + 2{K_\rho}^{z 2}\big] \\
&+ \big[{K_\rho}^{\rho} A_{\rho,\rho} + {K_\rho}^{z}\big(A_{z,\rho} + A_{\rho,z}\big) + {K_z}^{z} A_{z,z}\big] \\
&- A_\rho\big[\frac{s_{,\rho}}{s}\big({K_\rho}^{\rho} - \chi\big) + {K_\rho}^{z}\big(\frac{s_{,z}}{s} + \frac{2a_{,z}}{a}\big) + \frac{a_{,\rho}}{a}\big({K_\rho}^{\rho} - {K_z}^{z}\big)\big] \\
&- A_z\big[\frac{s_{,z}}{s}\big({K_z}^{z} - \chi\big) + {K_\rho}^{z}\big(\frac{s_{,\rho}}{s} + \frac{2a_{,\rho}}{a}\big) + \frac{a_{,z}}{a}\big({K_z}^{z} - {K_\rho}^{\rho}\big)\big] \\
&- iea^2\big(\phi^*\Pi_\phi - \phi{\Pi_\phi}^*\big) \\
&- 8\pi G\big[A_\rho\big(\Pi_\phi^*\mathcal{D}_\rho\phi + \Pi_\phi\big(\mathcal{D}_\rho\phi\big)^*\big) + A_z\big(\Pi_\phi^*\mathcal{D}_z\phi + \Pi_\phi\big(\mathcal{D}_z\phi\big)^*\big) + \frac{1}{a^2}\mathcal{B}_{\rho z}\big(\mathcal{E}_\rho A_z - \mathcal{E}_z A_\rho\big)\big]
\end{aligned}$$

The relevant Einstein equations are

$$\dot{s} - \beta^\rho s_{,\rho} - \beta^z s_{,z} = -\alpha s\chi$$

$$\dot{\chi} - \beta^\rho\chi_{,\rho} - \beta^z\chi_{,z} = -\frac{1}{sa^2}\left[(\alpha s_{,\rho})_{,\rho} + (\alpha s_{,z})_{,z}\right] - 4\pi G\frac{\alpha}{a^2}\left[2a^2\,V(\phi,\phi^*) - \frac{1}{a^2}{\mathcal{B}_{\rho z}}^2 + {\mathcal{E}_\rho}^2 + {\mathcal{E}_z}^2\right]$$

$$\begin{aligned}
(\log a^2)_{,\rho\rho} + (\log a^2)_{,zz} = {}& -\frac{2}{s}(s_{,\rho\rho} + s_{,zz}) - \frac{a^2}{2\alpha^2}\left[({\beta^\rho}_{,\rho} - {\beta^z}_{,z})^2 + ({\beta^z}_{,\rho} + {\beta^\rho}_{,z})^2\right] - \frac{3}{2}a^2\chi^2 \\
&- 16\pi G\left[a^2|\Pi_\phi|^2 + \left|\mathcal{D}_\rho\phi\right|^2 + \left|\mathcal{D}_z\phi\right|^2 + a^2\,V(\phi,\phi^*) + \frac{1}{2}\left({\mathcal{E}_\rho}^2 + {\mathcal{E}_z}^2 + \frac{1}{a^2}{\mathcal{B}_{\rho z}}^2\right)\right]
\end{aligned}$$

$$\begin{aligned}
{\beta^\rho}_{,\rho\rho} + {\beta^\rho}_{,zz} = {}& \Big(\log\frac{\alpha}{a^2 s}\Big)_{,\rho}({\beta^\rho}_{,\rho} - {\beta^z}_{,z}) + \Big(\log\frac{\alpha}{a^2 s}\Big)_{,z}({\beta^z}_{,\rho} + {\beta^\rho}_{,z}) + \alpha\chi_{,\rho} + 3\alpha\chi\frac{s_{,\rho}}{s} \\
&- 16\pi G\alpha\left[\mathcal{D}_\rho\phi\,{\Pi_\phi}^* + \text{c.c.} - \frac{1}{a^2}\mathcal{B}_{\rho z}\mathcal{E}_z\right]
\end{aligned}$$

$$\begin{aligned}
{\beta^z}_{,\rho\rho} + {\beta^z}_{,zz} = {}& \Big(\log\frac{\alpha}{a^2 s}\Big)_{,\rho}({\beta^z}_{,\rho} + {\beta^\rho}_{,z}) + \Big(\log\frac{\alpha}{a^2 s}\Big)_{,z}({\beta^z}_{,z} - {\beta^\rho}_{,\rho}) + \alpha\chi_{,z} + 3\alpha\chi\frac{s_{,z}}{s} \\
&- 16\pi G\alpha\left[\mathcal{D}_z\phi\,{\Pi_\phi}^* + \text{c.c.} + \frac{1}{a^2}\mathcal{B}_{\rho z}\mathcal{E}_z\right]
\end{aligned}$$

$$\begin{aligned}
\alpha_{,\rho\rho} + \alpha_{,zz} = {}& -\alpha\left[(\log a^2)_{,\rho\rho} + (\log a^2)_{,zz}\right] - \frac{2\alpha}{s}(s_{,\rho\rho} + s_{,zz}) - \frac{1}{s}(s_{,\rho}\alpha_{,\rho} + s_{,z}\alpha_{,z}) \\
&- 16\pi G\alpha\left[\left|\mathcal{D}_\rho\phi\right|^2 + \left|\mathcal{D}_z\phi\right|^2 + \frac{3}{2}a^2\,V(\phi,\phi^*) + \frac{1}{4}\big(\frac{1}{a^2}{\mathcal{B}_{\rho z}}^2 + {\mathcal{E}_\rho}^2 + {\mathcal{E}_z}^2\big)\right]
\end{aligned}$$

In these, we have used the auxiliary definitions

$$\begin{aligned}
\mathcal{E}_\rho &= \Pi_\rho + \big(nA\big)_{,\rho} + \frac{1}{a^2}{K_\rho}^{\rho}A_\rho + \frac{1}{a^2}{K_\rho}^{z}A_z \\
\mathcal{E}_z &= \Pi_z + \big(nA\big)_{,z} + \frac{1}{a^2}{K_z}^{\rho}A_\rho + \frac{1}{a^2}{K_z}^{z}A_z \\
\mathcal{B}_{\rho z} &= A_{z,\rho} - A_{\rho,z} \\
{K_\rho}^{\rho} &= \frac{1}{3\alpha}\left(-\alpha\rho\bar{\Omega} + {\beta^\rho}_{,\rho} - {\beta^z}_{,z}\right) \\
{K_z}^{z} &= \frac{1}{3\alpha}\left(-\alpha\rho\bar{\Omega} - 2{\beta^\rho}_{,\rho} + 2{\beta^z}_{,z}\right) \\
{K_\rho}^{z} &= \frac{1}{2\alpha}\left({\beta^z}_{,\rho} + {\beta^\rho}_{,z}\right) \\
\rho\bar{\Omega} &= \frac{1}{2\alpha}\left(3\alpha\chi - {\beta^\rho}_{,\rho} + {\beta^z}_{,z}\right)
\end{aligned}$$

All the (regularized) equations

After some experience, it has become clear that we must use somewhat different variables. The following equations use "regularized" variables. More particularly, they incorporate the substitutions $a = \psi^2$, $s = \rho\psi^2 e^{\rho\bar{\sigma}}$, and $\chi = \frac{1}{3\alpha}\big(2\alpha\rho\bar{\Omega} + \beta^\rho_{,\rho} - \beta^z_{,z}\big)$. In addition, we use the Hamiltonian constraint in the slicing equation which changes the latter to a nonlinear equation in α.

$$
\begin{aligned}
\big(nA\big)_{,t} &= \beta^\rho \big(nA\big)_{,\rho} + \beta^z \big(nA\big)_{,z} + \alpha\,\Pi_{(nA)} + \frac{1}{\psi^4}\big[\alpha_{,\rho}A_\rho + \alpha_{,z}A_z\big] \\
\Pi_{(nA),t} &= \beta^\rho \Pi_{(nA),\rho} + \beta^z \Pi_{(nA),z} - \frac{\alpha}{\psi^4}\big[\big(nA\big)_{,\rho\rho} + \big(nA\big)_{,zz}\big] + \frac{2}{\rho\psi^6 e^\sigma}\big[\big(\rho\alpha\psi^2 e^\sigma \Pi_\rho\big)_{,\rho} + \big(\rho\alpha\psi^2 e^\sigma \Pi_z\big)_{,z}\big] \\
&\quad + 2\alpha\big(nA\big)\big[{K_\rho}^{\rho 2} + {K_z}^{z 2} + {K_\rho}^{z 2} + {K_\rho}^\rho {K_z}^z\big] \\
&\quad - \frac{\alpha}{\psi^4}\big[\big(nA\big)_{,\rho}\frac{\big(\rho\psi^2 e^\sigma\big)_{,\rho}}{\rho\psi^2 e^\sigma} + \big(nA\big)_{,z}\frac{\big(\psi^2 e^\sigma\big)_{,z}}{\psi^2 e^\sigma} - \mathcal{E}_\rho\frac{\alpha_{,\rho}}{\alpha} - \mathcal{E}_z\frac{\alpha_{,z}}{\alpha}\big] \\
&\quad - ie\alpha\big(\phi^*\Pi_\phi - \phi\Pi^*_\phi\big) + 8\pi G\,\alpha\big(nA\big)\left[2\,\big|\Pi_\phi\,\big|^2 - V + \frac{1}{2\psi^4}\big({\mathcal{E}_\rho}^2 + {\mathcal{E}_z}^2 + \frac{1}{\psi^4}{\mathcal{B}_{\rho z}}^2\big)\right]
\end{aligned}
$$

$$
\begin{aligned}
A_{\rho,t} &= \beta^\rho A_{\rho,\rho} + \beta^z A_{\rho,z} - \big(nA\big)\alpha_{,\rho} - \alpha\Pi_\rho + A_\rho\,\big(-\alpha {K_\rho}^\rho + \beta^\rho_{,\rho}\big) + A_z\,\frac{1}{2}\big(\beta^z_{,\rho} - \beta^\rho_{,z}\big) \\
\Pi_{\rho,t} &= \big(\beta^\rho \Pi_\rho\big)_{,\rho} + \beta^z \Pi_{\rho,z} + \Pi_z {\beta^z}_{,\rho} + \alpha\big(\frac{1}{\psi^4}\mathcal{B}_{\rho z}\big)_{,z} - \frac{\alpha}{\rho\psi^2 e^\sigma}\big[\frac{\rho e^\sigma}{\psi^2}\big(A_{\rho,\rho} + A_{z,z}\big)\big]_{,\rho} \\
&\quad - \frac{1}{\rho\psi^6 e^\sigma}\big[\big(\rho\alpha\psi^2 e^\sigma\big)_{,\rho}\big(A_{\rho,\rho} + \frac{\big(\psi^2 e^\sigma\big)_{,z}}{\psi^2 e^\sigma}A_z\big) + \big(\rho\alpha\psi^2 e^\sigma\big)_{,z}\big(A_{\rho,z} - \frac{\big(\rho\psi^2 e^\sigma\big)_{,\rho}}{\rho\psi^2 e^\sigma}A_z\big)\big] \\
&\quad + A_\rho\frac{2}{\rho\psi^6 e^\sigma}\left[\big(\psi\psi_{,\rho}\alpha\rho e^\sigma\big)_{,\rho} + \big(\psi\psi_{,z}\alpha\rho e^\sigma\big)_{,z}\right] + \alpha\Pi_{(nA)}\big(\frac{\big(\rho\psi^2 e^\sigma\big)_{,\rho}}{\rho\psi^2 e^\sigma} - \frac{\alpha_{,\rho}}{\alpha}\big) - \alpha {K_\rho}^\rho \mathcal{E}_\rho - \alpha {K_\rho}^z \mathcal{E}_z \\
&\quad + ie\alpha\big[\phi^*\mathcal{D}_\rho\phi - \phi\big(\mathcal{D}_\rho\phi\big)^*\big] \\
&\quad + 8\pi G\frac{\alpha}{\psi^4}\Big\{2A_\rho\,\big|\mathcal{D}_\rho\phi\big|^2 + A_z\big[\mathcal{D}_\rho\phi\big(\mathcal{D}_z\phi\big)^* + \text{c.c.}\big] + A_\rho\,\psi^4 V - \psi^4\,\big(nA\big)\big[\mathcal{D}_\rho\phi\,\Pi^*_\phi + \text{c.c.}\big] \\
&\qquad + \frac{A_\rho}{2}\big(\frac{{\mathcal{B}_{\rho z}}^2}{\psi^4} - {\mathcal{E}_\rho}^2 + {\mathcal{E}_z}^2\big) - A_z\mathcal{E}_\rho\mathcal{E}_z + \big(nA\big)\mathcal{E}_z\mathcal{B}_{\rho z}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
A_{z,t} &= \beta^\rho A_{z,\rho} + \beta^z A_{z,z} - \big(nA\big)\alpha_{,z} - \alpha\Pi_z + A_\rho\,\frac{1}{2}\big(\beta^\rho_{,z} - \beta^z_{,\rho}\big) + A_z\,\big(-\alpha {K_\rho}^\rho + \beta^\rho_{,\rho}\big) \\
\Pi_{z,t} &= \beta^\rho \Pi_{z,\rho} + \big(\beta^z \Pi_z\big)_{,z} + \Pi_\rho\,{\beta^\rho}_{,z} - \alpha\big(\frac{1}{\psi^4}\mathcal{B}_{\rho z}\big)_{,\rho} - \frac{\alpha}{\rho\psi^2 e^\sigma}\big[\frac{\rho e^\sigma}{\psi^2}\big(A_{\rho,\rho} + A_{z,z}\big)\big]_{,z} \\
&\quad - \frac{1}{\rho\psi^6 e^\sigma}\big[\big(\rho\alpha\psi^2 e^\sigma\big)_{,\rho}\big(A_{z,\rho} - \frac{2\psi_{,z}}{\psi}A_z\big) + \big(\rho\alpha\psi^2 e^\sigma\big)_{,z}\big(A_{z,z} + \frac{2\psi_{,\rho}}{\psi}A_\rho\big)\big] \\
&\quad + A_z\frac{2}{\rho\psi^6 e^\sigma}\left[\big(\psi\psi_{,\rho}\rho\alpha e^\sigma\big)_{,\rho} + \big(\psi\psi_{,z}\rho\alpha e^\sigma\big)_{,z}\right] + \alpha\Pi_{(nA)}\big(\frac{\big(\rho\psi^2 e^\sigma\big)_{,z}}{\rho\psi^2 e^\sigma} - \frac{\alpha_{,z}}{\alpha}\big) - \alpha {K_z}^\rho \mathcal{E}_\rho - \alpha {K_z}^z \mathcal{E}_z \\
&\quad + ie\alpha\big[\phi^*\mathcal{D}_z\phi - \phi\big(\mathcal{D}_z\phi\big)^*\big] \\
&\quad + 8\pi G\frac{\alpha}{\psi^4}\Big\{2A_z\big|\mathcal{D}_z\phi\big|^2 + A_z\big[\mathcal{D}_\rho\phi\big(\mathcal{D}_z\phi\big)^* + \text{c.c.}\big] + A_z\,\psi^4\,V - \big(nA\big)\psi^4\big[\mathcal{D}_z\phi\,\Pi^*_\phi + \text{c.c.}\big] \\
&\qquad + A_z\big(\frac{{\mathcal{B}_{\rho z}}^2}{\psi^4} + {\mathcal{E}_\rho}^2 - {\mathcal{E}_z}^2\big) - A_\rho\mathcal{E}_\rho\mathcal{E}_z - \big(nA\big)\mathcal{E}_\rho\mathcal{B}_{\rho z}\Big\}
\end{aligned}
$$

$$\phi_{,t} = \beta^\rho \phi_{,\rho} + \beta^z \phi_{,z} + ie\alpha\big(nA\big)\phi + \alpha\Pi_\phi$$

$$\begin{aligned}\Pi_{\phi,t} = {} & \beta^\rho \Pi_{\phi,\rho} + \beta^z \Pi_{\phi,z} + ie\alpha\big(nA\big)\,\Pi_\phi - ie\alpha\big(nA\big)\phi\frac{1}{3}\big(2\alpha\Omega + \beta^\rho{}_{,\rho} - \beta^z{}_{,z}\big) - \alpha\frac{\partial V}{\partial \phi^*} \\ & + \frac{\alpha}{\psi^4}\big(\phi_{,\rho\rho} + \phi_{,zz}\big) - ie\frac{\alpha}{\psi^4}\left[\big(\phi A_\rho\big)_{,\rho} + \big(\phi A_z\big)_{,z}\right] \\ & + \frac{\alpha}{\psi^4}\left[\left(\frac{(\rho\alpha\psi e^\sigma)_{,\rho}}{\rho\alpha\psi e^\sigma} - ieA_\rho\phi\right)\big(\phi_{,\rho} - ieA_\rho\phi\big) + \left(\frac{(\alpha\psi e^\sigma)_{,z}}{\alpha\psi e^\sigma} - ieA_z\phi\right)\big(\phi_{,z} - ieA_z\phi\big)\right]\end{aligned}$$

The constraint equations become

$$0 = \frac{1}{\psi^4}\big(A_{\rho,\rho} + A_{z,z}\big) - \Pi_{(nA)} + \frac{1}{\rho\psi^6 e^\sigma}\big(A_\rho\big(\rho\psi^2 e^\sigma\big)_{,\rho} + A_z\big(\rho\psi^2 e^\sigma\big)_{,z}\big) + \frac{(nA)}{3\alpha}\big(2\alpha\Omega + \beta^\rho{}_{,\rho} - \beta^z{}_{,z}\big)$$

$$0 = \big(\rho\psi^2 e^\sigma \mathcal{E}_\rho\big)_{,\rho} + \big(\rho\psi^2 e^\sigma \mathcal{E}_\rho\big)_{,\rho} - ie\big(\psi^6\rho e^\sigma\big)\big(\phi\Pi^*_\phi - \phi^*\Pi_\phi\big)$$

The relevant Einstein equations are

$$\dot{\bar{\sigma}} = 2\beta^\rho\,(\rho\bar{\sigma})_{,\rho^2} + \beta^z\bar{\sigma}_{,z} - \left(\alpha\bar{\Omega} + \left[\frac{\beta^\rho}{\rho}\right]_{,\rho}\right)$$

$$\begin{aligned}\dot{\bar{\Omega}} = {} & 2\beta^\rho\left(\rho\bar{\Omega}\right)_{,\rho} + \beta^z\bar{\Omega}_{,z} + \frac{1}{2\alpha\rho}\left(\beta^\rho{}_{,z}{}^2 - \beta^z{}_{,\rho}{}^2\right) \\ & + \frac{1}{\psi^4}\left(\frac{\alpha_{,\rho}}{\rho}\right)_{,\rho} + \frac{\alpha}{\psi^6}\left(\frac{\psi^2{}_{,\rho}}{\rho}\right)_{,\rho} - \frac{\alpha}{\rho\psi^4}\cdot\frac{(\psi^4 e^{\rho\bar{\sigma}})_{,\rho}}{\psi^4 e^{\rho\bar{\sigma}}}\cdot\big(\log(\alpha\psi^2)\big)_{,\rho} \\ & - \frac{\alpha}{\psi^4}\bar{\sigma}_{,z}\left(\log(\alpha\psi^4)\right)_{,z} - \frac{\alpha}{\psi^4}\left(\rho\bar{\sigma}_{,z}{}^2 + \bar{\sigma}_{,zz}\right) + 64\pi\frac{\alpha}{\psi^4}\rho(\phi_{,\rho^2})^2\end{aligned}$$

$$\begin{aligned}-16\pi\left(\Pi^2 + \phi_{,\rho}{}^2 + \phi_{,z}{}^2\right) = {} & 8\frac{\psi_{,\rho\rho}}{\psi} + 8\frac{\psi_{,zz}}{\psi} + \frac{2}{e^{\rho\bar{\sigma}}}\left\{\frac{1}{\rho^2}\left(\rho^2(e^{\rho\bar{\sigma}})_{,\rho}\right)_{,\rho} + (e^{\rho\bar{\sigma}})_{,zz}\right\} \\ & + \frac{8}{\psi e^{\rho\bar{\sigma}}}\left[\frac{1}{\rho}(\rho e^{\rho\bar{\sigma}})_{,\rho}\psi_{,\rho} + (e^{\rho\bar{\sigma}})_{,z}\psi_{,z}\right] \\ & + \frac{a^2}{2\alpha^2}\left[(\beta^\rho{}_{,\rho} - \beta^z{}_{,z})^2 + (\beta^z{}_{,\rho} + \beta^\rho{}_{,z})^2\right] \\ & + \frac{a^2}{6\alpha^2}\left[2\alpha\rho\bar{\Omega} + \beta^\rho{}_{,\rho} - \beta^z{}_{,z}\right]^2\end{aligned}$$

$$\begin{aligned}0 = {} & \frac{2}{3}\beta^\rho{}_{,\rho\rho} + \beta^\rho{}_{,zz} + \frac{1}{3}\beta^z{}_{,z\rho} + \left(\log\frac{\psi^6 e^{\rho\bar{\sigma}}}{\alpha}\right)_{,z}\left[\beta^z{}_{,\rho} + \beta^\rho{}_{,z}\right] \\ & - \frac{2}{3}\left(\log\frac{\psi^6}{\alpha}\right)_{,\rho}\left[\beta^z{}_{,z} - \beta^\rho{}_{,\rho}\right] - \frac{2}{3}\frac{\alpha\rho}{\psi^6 e^{3\rho\bar{\sigma}}}\left(\psi^6 e^{3\rho\bar{\sigma}}\bar{\Omega}\right)_{,\rho} - \frac{8}{3}\alpha\bar{\Omega} + 32\pi\frac{\alpha}{\psi^2}\Pi\phi_{,\rho}\end{aligned}$$

$$\begin{aligned}0 = {} & \beta^z{}_{,\rho\rho} + \frac{4}{3}\beta^z{}_{,zz} - \frac{1}{3}\beta^\rho{}_{,z\rho} + \frac{2\alpha}{\psi^6 e^{\rho\bar{\sigma}}}\left(\rho\frac{\psi^6 e^{\rho\bar{\sigma}}}{\alpha}\right)_{,\rho^2}\left[\beta^z{}_{,\rho} + \beta^\rho{}_{,z}\right] \\ & + \frac{4}{3}\left(\log\frac{\psi^6 e^{3\rho\bar{\sigma}/2}}{\alpha}\right)_{,z}\left[\beta^z{}_{,z} - \beta^\rho{}_{,\rho}\right] - \frac{2}{3}\frac{\alpha\rho}{\psi^6}\left(\psi^6\bar{\Omega}\right)_{,z} + 32\pi\frac{\alpha}{\psi^2}\Pi\phi_{,z}\end{aligned}$$

$$\begin{aligned}0 = {} & 2\left(\rho\alpha_{,\rho}\right)_{,\rho^2} + \alpha_{,zz} + \frac{1}{\psi^2 e^{\rho\bar{\sigma}}}\left(\alpha_{,\rho}(\psi^2 e^{\rho\bar{\sigma}})_{,\rho} + \alpha_{,z}(\psi^2 e^{\rho\bar{\sigma}})_{,z}\right) \\ & - \frac{\psi^4}{2\alpha}\left[(\beta^\rho{}_{,\rho} - \beta^z{}_{,z})^2 + (\beta^z{}_{,\rho} + \beta^\rho{}_{,z})^2\right] \\ & - \frac{\psi^4}{6\alpha}\left[2\alpha\rho\bar{\Omega} + \beta^\rho{}_{,\rho} - \beta^z{}_{,z}\right]^2 - 16\pi\alpha\Pi^2\end{aligned}$$

The gauge condition can be written as an equation for $\Pi_{(nA)}$ at the initial time

$$\Pi_{(nA)} = \frac{1}{a^2}\big(A_{\rho,\rho} + A_{z,z}\big) + \frac{1}{sa^2}\big(A_\rho s_{,\rho} + A_z s_{,z}\big) + \big(nA\big)\chi$$

In these, we have the auxiliary definitions

$$\begin{aligned}
\mathcal{E}_\rho &= \Pi_\rho + \big(nA\big)_{,\rho} + \frac{1}{a^2}K_\rho{}^\rho A_\rho + \frac{1}{a^2}K_\rho{}^z A_z \\
\mathcal{E}_z &= \Pi_z + \big(nA\big)_{,z} + \frac{1}{a^2}K_z{}^\rho A_\rho + \frac{1}{a^2}K_z{}^z A_z \\
K_\rho{}^\rho &= \frac{1}{3\alpha}\left(-\alpha\rho\bar{\Omega} + \beta^\rho{}_{,\rho} - \beta^z{}_{,z}\right) \\
K_z{}^z &= \frac{1}{3\alpha}\left(-\alpha\rho\bar{\Omega} - 2\beta^\rho{}_{,\rho} + 2\beta^z{}_{,z}\right) \\
K_\rho{}^z &= \frac{1}{2\alpha}\left(\beta^z{}_{,\rho} + \beta^\rho{}_{,z}\right) \\
\rho\bar{\Omega} &= \frac{1}{2\alpha}\left(3\alpha\chi - \beta^\rho{}_{,\rho} + \beta^z{}_{,z}\right)
\end{aligned}$$