

## “Preliminaries”

Our purpose is to combine the classical Yang-Mills fields with Einstein gravity and examine the evolution of such fields in spherical symmetry. To this end we of course need the equations of motion. There are a variety of conventions in the literature and to be honest I have yet to sort it all out. Occasionally, I can't tell if a difference is due to a convention or a typo. At the risk of being wrong, what follows is my best understanding to date of the equations for the Einstein-Yang-Mills-Higgs theory. Of course all this should be double checked.

Let's start with some definitions and conventions. We want to work with an  $SU(2)$  Yang-Mills field. Other gauge groups would of course be possible, but  $SU(2)$  has been studied extensively and a lot of results have been found in this case, such as the t'Hooft-Polyakov monopole in flat space, the Bartnik-McKinnon particle solutions, “colored” black holes, and Matt's original work in Yang-Mills collapse. Another somewhat more practical reason is that we have a simple ansatz for the form of the gauge connection  $A_\mu^a$  in spherically symmetric  $SU(2)$ . To go beyond this gauge group would require finding a similar ansatz. A similar thing would be true if we were to generalize our model from spherical symmetry to axisymmetry. We would need the appropriate form for  $A_\mu^a$ .

Okay, let's start by trying to unravel Witten's original ansatz for the gauge field. Most people have referred to Witten's 1977 PRL so let's try to understand it. His ansatz is as follows:

$$A_j^a = \frac{\phi_2 + 1}{r^2} \epsilon_{jak} x_k + \frac{\phi_1}{r^3} (\delta_{ja} r^2 - x_j x_a) + A_1 \frac{x_j x_a}{r^2}$$

$$A_0^a = \frac{A_0 x^a}{r}.$$

where the functions  $A_0, A_1, \phi_0$ , and  $\phi_1$  are all functions of only  $r$  and  $t$ . Note too that we will shortly change Witten's original functions to match those which seem to appear most commonly in the EYM literature. The gauge indices  $a$  run over 1,2, and 3 as do the spacetime indices  $i, j$  and  $k$ . Both sets of indices refer to Cartesian coordinates in both the group space and spacetime (e.g.  $x_1 = x, x_2 = y$ , etc). Witten is of course working in flat space here. The matrices that would be attached to the  $A_\mu^a$  to make the full  $A_\mu$  are the Pauli matrices up to some normalization. We will call these matrices  $\tau_1, \tau_2$  and  $\tau_3$ . Note that these will *not* be the same as those used by say Zhou or BM. Witten's ansatz “mixes” group and spacetime aspects. For that reason it is important to get the correct parameterization and that was a source of confusion for me for a time. The parameterization that several authors use in discussing these solutions in GR (Bartnik and McKinnon, Zhou and Straumann, Breitenlohner, etc) uses a different parameterization better suited to their assumption of a spherically symmetric metric. The analogy between Cartesian unit vectors and spherical polar unit vectors is appropriate here. The relation between the two is given by

$$\tau_1 = \tau_r \sin \theta \cos \phi + \tau_\theta \cos \theta \cos \phi - \tau_\phi \sin \phi$$

$$\tau_2 = \tau_r \sin \theta \sin \phi + \tau_\theta \cos \theta \sin \phi + \tau_\phi \cos \phi$$

$$\tau_3 = \tau_r \cos \theta - \tau_\theta \sin \theta$$

with the usual expression for the inverse relation. One can think of this choice for the basis of  $SU(2)$  as the spherical projection of the Pauli matrices (Ershov and Galtsov). Bartnik and McKinnon I guess are trying to say as much in their rather enigmatic paragraph where they write down their form for the gauge field  $A$ . By substituting these expressions into Witten's ansatz and making the change of variables from  $(x, y, z)$  to  $(r, \theta, \phi)$ , one arrives at the following expression for the gauge connection  $A$

$$A = A_0 \tau_r dt + A_1 \tau_r dr + (\phi_1 \tau_\theta + (\phi_2 + 1) \tau_\phi) d\theta + (\phi_1 \tau_\phi - (\phi_2 + 1) \tau_\theta) \sin \theta d\phi.$$

The algebra to get to this point is, as they say, tedious, but I have done it and verified that it is indeed the case. Examining this, however, you will note that this is not exactly what most people have used. This form differs from the others by a gauge transformation. (I need to give this transformation... It is included in Greene, Mathur and O'Neill)

As I mentioned, in this we will use  $\tau_r, \tau_\theta$  and  $\tau_\phi$ . They are what BM and Zhou call  $\tau_3, \tau_1$  and  $\tau_2$  respectively. Note that both sets satisfy the usual commutation relations with the same choice of normalization.

Why am I saying so much about this? Part of it is simply talking to myself so that I know up front what our conventions are. But it turns out that understanding this (at least for me) was crucial in understanding the extension to a Higgs field and the corresponding ansatz for it in spherical symmetry.

In our current case, the gauge connection in a form notation is (Zhou)

$$A = u\tau_r dt + v\tau_r dr + (w\tau_\theta + \tilde{w}\tau_\phi)d\theta + (\cot\theta\tau_r + w\tau_\phi - \tilde{w}\tau_\theta)\sin\theta d\phi.$$

where the functions  $u, v, w$ , and  $\tilde{w}$  are all functions of  $r$  and  $t$ . The  $\tau_i$ 's (with  $i \in \{r, \theta, \phi\}$ ) are the basis of the group  $SU(2)$ . They have commutation relations which we will take to be

$$[\tau_i, \tau_j] = \bar{\beta}\epsilon_{ijk}\tau_k$$

where  $\bar{\beta}$  is essentially a choice of normalization. The trace of the square of these matrices is then<sup>1</sup>

$$Tr(\tau_i^2) = -\frac{\bar{\beta}^2}{2}.$$

One frequent choice in high energy physics is  $\bar{\beta} = i$  in which case the  $\tau_i$ 's are just one half the spherical projection of the usual Pauli matrices.<sup>2</sup> Several authors that I have read (Ginsparg, Xanthopoulos, Harvey) use anti-Hermitian generators of  $SU(2)$  which would seem to correspond to a choice of  $\bar{\beta} = 1$ . In this case, the  $\tau_i$ 's are  $-i/2$  times our spherical projection of the Pauli matrices and obey  $(\tau_i)^\dagger = -\tau_i$ . This is actually the choice we will make eventually, but we will allow for  $\bar{\beta}$  to be general for the moment.

In any case, most people make some simplifications to  $A$ . Several authors note that we can set the function  $v$  to be zero by gauge invariance. Specifically, the form of  $A$  that we have after assuming spherical symmetry and this particular parameterization still has some gauge freedom. A general gauge transformation is of the form

$$A \rightarrow A' = U^{-1}AU + U^{-1}d(U)$$

where the transformation matrix  $U$  is unitary. For a general transformation,  $A$  will change, but physical quantities will not change. However, a subset of these general transformations will leave the form of  $A$  unchanged. This is, of course, analogous to GR. In spherical symmetry, we can set the shift vector to zero (a choice of gauge). Likewise in the YM case, a particular gauge choice is  $v = 0$ . The gauge freedom still present in the original connection is described by a  $U(1)$  group. More particularly, we have the freedom to choose  $U(x) = e^{\psi(t,r)\tau_r}$  and our gauge connection will maintain its current form.<sup>3</sup> When one works out the form of the above gauge transformation the connection becomes

$$\begin{aligned} A \rightarrow A' &= (u + \psi_{,t})\tau_r dt + (v + \psi_{,r})\tau_r dr \\ &+ [(w \cos \psi + \tilde{w} \sin \psi)\tau_\theta + (\tilde{w} \cos \psi - w \sin \psi)\tau_\phi] d\theta \\ &+ [\cot \theta \tau_r + (w \cos \psi + \tilde{w} \sin \psi)\tau_\phi - (\tilde{w} \cos \psi - w \sin \psi)\tau_\theta] \sin \theta d\phi \end{aligned}$$

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<sup>1</sup> Note that this helps explain the form the Lagrangian takes in some of the literature:

$$\mathcal{L}_{YM} = -\frac{1}{g^2} F_{\mu\nu}^a F^{a\mu\nu} = \frac{2}{g^2 \bar{\beta}^2} Tr(\mathbf{F}_{\mu\nu} \mathbf{F}^{\mu\nu})$$

If  $\bar{\beta}$  is imaginary, we have the usual sign, but if it is real, then in the Lagrangian, the ostensible sign seems naively to be “wrong” since it is actually taken up in the trace over the matrices. We will ultimately be more interested in a component form of the  $F$ 's in group space so we will not worry about this very minor technicality.

<sup>2</sup> Did you follow that? If we had been using the  $\tau_1$ , etc of say, Witten, then these  $\tau$ 's would be one half the usual Pauli matrices. We are using a parameterization of these matrices appropriate to spherical symmetry, but the commutation relations and normalization are unchanged.

<sup>3</sup> Note here that  $\psi(t, r)$  is imaginary if our generators are Hermitian ( $\bar{\beta}$  is imaginary) and real if our generators are anti-Hermitian ( $\bar{\beta}$  is real).

We can now choose to work in a gauge then such that  $\psi_{,r} + v = 0$ . So we simply set  $v = 0$ .<sup>4</sup>

More mysterious perhaps is the claim that  $u(r, t)$  can be set to zero as well. Most people just put it to zero saying they want a purely magnetic connection meaning they are looking only for magnetic monopoles and are disregarding the possibility of dyons (solutions with magnetic and electric charge). Zhou shows, however, that in the absence of matter, the requirements of asymptotic flatness and finite energy density constrain  $u$  to be zero. I will not reproduce his argument here although I will mention that we should just double check it in the context of our more general parameterization of the spherically symmetric metric since Zhou works only in radial gauge. Zhou also makes a rather enigmatic comment by saying that  $\tilde{w}$  can be set to zero as well since it appears symmetrically in the equations. To be honest, I'm not quite sure what he means by that. The fact as I understand it is that in the static case, if you keep  $\tilde{w}$  as part of the gauge connection and derive the equations with it, you end up with a simple equation which when solved gives you  $\tilde{w} = Cw$  with  $C$  a constant. Now consider a constant gauge transformation:

$$U = \exp(\gamma\tau_r)$$

where  $\gamma$  is just a constant. This gauge transformation can now be performed to eliminate the proportionality constant  $C$ . This is, of course, the same as setting  $\tilde{w}$  to zero. We will see what happens in the non-static case below.

With  $A$  in hand we can begin calculating various derived quantities. The field strength tensor is defined by

$$\begin{aligned} F &= DA \\ &= dA + \bar{\alpha}A \wedge A. \end{aligned}$$

where  $D$  signifies here the gauge covariant derivative and where I am using the constant  $\bar{\alpha}$  to cover a range of conventions. It is worthwhile mentioning that in component notation this becomes

$$F_{\mu\nu}dx^\mu dx^\nu = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu + \bar{\alpha}[A_\mu, A_\nu])dx^\mu dx^\nu.$$

Note that this is one half of the ‘‘usual’’ definition from high energy folks for  $F_{\mu\nu}$  and is reflected in our Lagrangian by an extra factor of 4.

For the moment, let us not make any restrictions to the form for  $A$  that we previously wrote down. We may make some in the future, but as we are interested in the general spherically symmetric ansatz for the spacetime metric, let us consider the equations of motion with a general spherically symmetric ansatz for the Yang-Mills gauge field. For our  $A$  then, we have

$$\begin{aligned} F &= \tau_r(\dot{v} - u')dt \wedge dr \\ &+ [(\dot{w} - \bar{\alpha}\bar{\beta}u\tilde{w})dt + (w' - \bar{\alpha}\bar{\beta}v\tilde{w})dr] \wedge \Omega_1 \\ &+ [(\dot{\tilde{w}} + \bar{\alpha}\bar{\beta}uw)dt + (\tilde{w}' + \bar{\alpha}\bar{\beta}vw)dr] \wedge \Omega_2 \\ &+ [(-1 + \bar{\alpha}\bar{\beta}(w^2 + \tilde{w}^2))\sin\theta\tau_r + (w\tau_\phi - \tilde{w}\tau_\theta)(1 - \bar{\alpha}\bar{\beta})\cos\theta] d\theta \wedge d\phi \end{aligned}$$

where we have defined

$$\begin{aligned} \Omega_1 &= (\tau_\theta d\theta + \tau_\phi \sin\theta d\phi) \\ \Omega_2 &= (\tau_\phi d\theta - \tau_\theta \sin\theta d\phi) \end{aligned}$$

We can take an additional exterior derivative of our equation for  $F$  and get

$$\begin{aligned} dF &= ddA + \bar{\alpha}(dA \wedge A - A \wedge dA) \\ &= \bar{\alpha}((F - \bar{\alpha}A \wedge A) \wedge A - A \wedge (F - \bar{\alpha}A \wedge A)) \\ &= \bar{\alpha}(F \wedge A - A \wedge F). \end{aligned}$$

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<sup>4</sup> This leaves some residual gauge freedom such that we can perform a gauge transformation where  $U(x) = e^{\psi(t)\tau_r}$ . We will come back to this somewhat later and in the problem we will ultimately consider fix the gauge completely.

This is an identity (*i.e.* a constraint equation) and not strictly an equation of motion. However, as in E&M there is a duality transformation which inverts the equations of motion and the constraints. Likewise in this case, we can simply make the change  $F \leftrightarrow *F$  everywhere and the resulting equation is our equation of motion in the absence of matter sources. If we consider this case, and unravel these equations in terms of their components, we get four equations.<sup>4</sup> We give two of these equations later with matter present. However, there are two that without matter are true only if we have the relation

$$\bar{\alpha}\bar{\beta} = 1$$

which of course constrains our normalization choices. From here on out we will make the simple choice that  $\bar{\alpha} = \bar{\beta} = 1$ . This corresponds to using anti-Hermitian matrices for the basis of  $SU(2)$ .

### The meat of the matter

Everything up to this point has been somewhat *ad hoc*. We have yet to refer to a Lagrangian for instance. Let's do that now. But let us also generalize from Einstein-Yang-Mills (EYM) to including some Higgs fields (EYMH). We thus start with a matter Lagrangian of the following form.

$$\mathcal{L} = \frac{1}{16\pi G}R - \frac{1}{g^2}F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2}D_\mu\Phi^a D^\mu\Phi^a - V(\Phi^a).$$

We are working with the gauge group  $SU(2)$ . The Higgs fields are in the so-called adjoint representation of  $SU(2)$ . This means simply that in general there are three Higgs fields and we denote this by letting the index  $a$  run over 1, 2 and 3.<sup>5,6</sup>

Again, the fields are defined as

$$F_{\mu\nu}^a = \frac{1}{2}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc}A_\mu^b A_\nu^c)$$

and the potential for the Higgs field is the ‘‘Mexican hat’’ potential

$$V(\Phi) = \frac{\lambda}{4}(\Phi^a\Phi^a - \eta^2)^2$$

The form of the gauge covariant derivative can be seen through its action on the Higgs field

$$D_\mu\Phi^a = \nabla_\mu\Phi^a + \epsilon^{abc}A_\mu^b\Phi^c$$

where  $\nabla_\mu$  is the covariant derivative associated with our metric  $g_{\mu\nu}$ . On varying the Lagrangian, the equations of motion for the matter are found to be<sup>7</sup>

$$D_\mu F^{a\mu\nu} = \frac{1}{2}g^2\epsilon^{abc}\Phi^b D^\nu\Phi^c$$

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<sup>4</sup> Recall that the dual of  $F_{\mu\nu}$  is

$$(*F)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}\sqrt{-g}F^{\sigma\rho}$$

where  $\epsilon_{tr\theta\phi} = 1$ .

<sup>5</sup> Other ways this is described is that the scalar fields are in the triplet of  $SU(2)$  or that they are in the **3** of  $SU(2)$ . Another possible model to consider would be that of a pair of complex scalar (Higgs) fields in the fundamental (doublet or **2**) representation of  $SU(2)$ . I believe this has some relation to the Einstein-Skyrme model, but I haven't looked at this yet.

<sup>6</sup> On a matter of notation, I use  $\Phi^a$  as the Higgs field. I will also use  $\Phi$  later as an auxiliary variable of the Yang-Mills equations which has no relation to the Higgs field per se. This is an attempt to have the equations look as much like Matt's as possible. To distinguish them, I will always try to add the group index if I am referring to the Higgs field. The context should also help distinguish the two.

<sup>7</sup> You will note that I have started with a differential forms approach and have now somewhat surreptitiously switched to using a coordinate approach. In some way this writeup is something of a chronology of how I worked on this problem. Once I got to working with the Higgs fields, it just seemed the coordinate approach was easier if for no other reason than everyone else seems to do the same. In passing, we could write the equation of motion for the gauge fields as  $D * F = d * F - *F \wedge A + A \wedge *F = J$  where  $D$  is here the gauge covariant derivative defined above and  $J$  is the appropriate current.

$$\begin{aligned}
D^\mu D_\mu \Phi^a &= \frac{\partial V}{\partial \Phi} \\
&= \lambda \Phi^a (\Phi^b \Phi^b - \eta^2)
\end{aligned}$$

Now consider the stress tensor. It is a natural generalization of the E&M stress tensor.

$$\begin{aligned}
T_{\mu\nu} &= -\frac{1}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} \\
&= \frac{1}{g^2} (2F_{\mu\lambda}^a F^a{}_{\nu}{}^\lambda - \frac{1}{2} g_{\mu\nu} F_{\alpha\beta}^a F^{a\alpha\beta}) + \frac{1}{2} (D_\mu \Phi^a D_\nu \Phi^a - \frac{1}{2} g_{\mu\nu} (D_\lambda \Phi^a D^\lambda \Phi^a)) - \frac{1}{2} g_{\mu\nu} V(\Phi)
\end{aligned}$$

We are finally ready to start doing some things closer to GR. We want to consider a general spherically symmetric metric. So we have as our line element

$$ds^2 = (-\alpha^2 + a^2 \beta^2) dt^2 + 2a^2 \beta dt dr + a^2 dr^2 + r^2 b^2 d\Omega^2$$

where the metric functions  $\alpha, a, \beta$  and  $b$  are functions of  $r$  and  $t$  and  $d\Omega^2$  is the metric on the unit sphere.

“Borrowing” wholesale from the appendix in Robert and Matt’s paper on black hole-scalar field interactions, we know in spherical symmetry that the three-metric  $h_{ij}$  and the extrinsic curvature tensor  $K^i{}_j$  are diagonal. We have

$$\begin{aligned}
h_{ij} &= \text{diag} (a^2(t, r), r^2 b^2(t, r), r^2 b^2 \sin^2 \theta) \\
K^i{}_j &= \text{diag} (K^r{}_r(t, r), K^\theta{}_\theta(t, r), K^\theta{}_\theta)
\end{aligned}$$

The nonzero components of the Christoffel symbols with respect to the 3-metric  $h_{ij}$  are:

$$\begin{aligned}
\Gamma^r{}_{rr} &= \frac{\partial_r a}{a} & \Gamma^r{}_{\theta\theta} &= -\frac{rb \partial_r (rb)}{a^2} & \Gamma^\theta{}_{r\theta} &= \frac{\partial_r (rb)}{rb} \\
\Gamma^r{}_{\phi\phi} &= -\sin^2 \theta \frac{rb \partial_r (rb)}{a^2} & \Gamma^\phi{}_{r\phi} &= \frac{\partial_r (rb)}{rb} \\
\Gamma^\theta{}_{\phi\phi} &= -\sin \theta \cos \theta & \Gamma^\phi{}_{\phi\theta} &= -\cot \theta
\end{aligned}$$

The two non-zero components of the Ricci tensor are

$$\begin{aligned}
R^r{}_r &= -\frac{2}{arb} \partial_r \left( \frac{\partial_r (rb)}{a} \right) \\
R^\theta{}_\theta &= \frac{1}{ar^2 b^2} \left[ a - \partial_r \left( \frac{rb}{a} \partial_r (rb) \right) \right].
\end{aligned}$$

The evolution equations for the metric components are

$$\begin{aligned}
\dot{a} &= -a\alpha K^r{}_r + (a\beta)' \\
\dot{b} &= -\alpha b K^\theta{}_\theta + \frac{\beta}{r} (rb)'.
\end{aligned}$$

The evolution equations for the components of the extrinsic curvature are

$$\begin{aligned}
\dot{K}^r{}_r &= \beta K^{r'}{}_r + \alpha K^r{}_r K - \frac{1}{a} \left( \frac{\alpha'}{a} \right)' - \frac{2\alpha}{arb} \left[ \frac{(rb)'}{a} \right]' + 4\pi G\alpha [S - \rho - 2S^r{}_r] \\
\dot{K}^\theta{}_\theta &= \beta K^{\theta'}{}_\theta + \alpha K^\theta{}_\theta K + \frac{\alpha}{(rb)^2} - \frac{1}{a(rb)^2} \left( \frac{\alpha rb}{a} (rb)' \right)' + 4\pi G\alpha [S - \rho - 2S^\theta{}_\theta]
\end{aligned}$$

where we define the spatial stress tensor as  $S_{ij} = T_{ij}$  and its trace as  $S = T_{ij}h^{ij}$ . We also define the energy density  $\rho = n_\mu n_\nu T^{\mu\nu}$  and the 3-momentum density  $j_i = -n_\mu T^\mu{}_i$ . The Hamiltonian constraint is

$$-\frac{2}{arb} \left[ \left( \frac{(rb)'}{a} \right)' + \frac{1}{rb} \left( \left( \frac{rb}{a} (rb)' \right)' - a \right) \right] + 4K^r{}_r K^\theta{}_\theta + 2K^{\theta 2}{}_\theta = 16\pi G\rho$$

and the momentum constraint is

$$-\frac{(rb)'}{rb} (K^\theta{}_\theta - K^r{}_r) - K^{\theta'}{}_\theta = 4\pi G j_r$$

Before writing down the Yang-Mills equations, let's make some convenient definitions:

$$\begin{aligned} \Pi &= \frac{a}{\alpha} [\dot{w} - u\tilde{w} - \beta(w' - v\tilde{w})] \\ \Phi &= w' - v\tilde{w} \\ P &= \frac{a}{\alpha} [\dot{\tilde{w}} + uw - \beta(\tilde{w}' + vw)] \\ Q &= \tilde{w}' + vw \end{aligned}$$

where  $\Pi, \Phi, P$  and  $Q$  are auxiliary variables which are intended to "eliminate" the first space and time derivatives of  $w$  and  $\tilde{w}$ .<sup>8</sup> To see this most easily, we can write

$$\begin{pmatrix} \Pi \\ \Phi \end{pmatrix} = M \begin{pmatrix} \dot{w} - u\tilde{w} \\ w' - v\tilde{w} \end{pmatrix} \\ \begin{pmatrix} P \\ Q \end{pmatrix} = M \begin{pmatrix} \dot{\tilde{w}} + uw \\ \tilde{w}' + vw \end{pmatrix}$$

where the matrix  $M$  is given by

$$M = \begin{pmatrix} \frac{a}{\alpha} & -\beta\frac{a}{\alpha} \\ 0 & 1 \end{pmatrix}$$

The inverse relations are then

$$\begin{pmatrix} \dot{w} \\ w' \end{pmatrix} = M^{-1} \begin{pmatrix} \Pi \\ \Phi \end{pmatrix} + \tilde{w} \begin{pmatrix} u \\ v \end{pmatrix} \\ \begin{pmatrix} \dot{\tilde{w}} \\ \tilde{w}' \end{pmatrix} = M^{-1} \begin{pmatrix} P \\ Q \end{pmatrix} - w \begin{pmatrix} u \\ v \end{pmatrix}$$

The full equations of motion for the Yang-Mills fields with the Higgs field can be written as follows

$$\begin{aligned} -\dot{\Pi} + \left[ \beta\Pi + \frac{\alpha}{a}\Phi \right]' + uP - v \left( \beta P + \frac{\alpha}{a}Q \right) + \frac{\alpha a}{b^2 r^2} w(1 - w^2 - \tilde{w}^2) &= g^2 \alpha a w H^2 \\ -\dot{P} + \left[ \beta P + \frac{\alpha}{a}Q \right]' - u\Pi + v \left( \beta\Pi + \frac{\alpha}{a}\Phi \right) + \frac{\alpha a}{b^2 r^2} \tilde{w}(1 - w^2 - \tilde{w}^2) &= g^2 \alpha a \tilde{w} H^2 \\ \left( \frac{b^2 r^2}{2\alpha a} (\dot{v} - u') \right)' &= \tilde{w}\Pi - wP \\ \left( \frac{b^2 r^2}{2\alpha a} (\dot{v} - u') \right)' &= \frac{\alpha}{a} (\tilde{w}\Phi - wQ) + \beta(\tilde{w}\Pi - wP). \end{aligned}$$

There are two consistency equations which come with our definitions of the auxiliary variables. Namely, the relations  $(\dot{w})' = (w')\dot{\phantom{w}}$  and  $(\dot{\tilde{w}})' = (\tilde{w}')\dot{\phantom{\tilde{w}}}$  imply

$$\begin{aligned} \dot{\Phi} &= \left[ \frac{\alpha}{a}\Pi + \beta\Phi \right]' + uQ - v \left[ \frac{\alpha}{a}P + \beta Q \right] - \tilde{w}(\dot{v} - u') \\ \dot{Q} &= \left[ \frac{\alpha}{a}P + \beta Q \right]' - u\Phi + v \left[ \frac{\alpha}{a}\Pi + \beta\Phi \right] + w(\dot{v} - u'). \end{aligned}$$

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<sup>8</sup> Note that as this progresses, I have tried to generalize the auxiliary variables from those Matt uses in his Yang-Mills work.

Turning now to the equation of motion for the Higgs field, we have

$$D^\mu D_\mu \Phi^a = \lambda \Phi^a (\Phi^b \Phi^b - \eta^2)$$

We want to use the following ansatz for the Higgs field compatible with our assumption of spherical symmetry

$$\Phi^a \tau_a = \tau_r H(t, r)$$

or that there is only one component to the Higgs field and it points in the “ $r$ ” direction of the internal space. The function  $H$  is a function of both  $t$  and  $r$ . Using this ansatz and substituting into the Higgs equation, we have three components. Two of these components (the  $\theta$  and  $\phi$  components) are trivially satisfied while the  $r$  component is the equation

$$\nabla^\mu \nabla_\mu H = \frac{2H}{r^2 b^2} (w^2 + \tilde{w}^2) + \lambda H (H^2 - \eta^2)$$

where  $\nabla_\mu$  is the derivative operator on the full spacetime manifold. We introduce the two auxiliary variables

$$\begin{aligned} C &= H' \\ D &= \frac{a}{\alpha} (\dot{H} - \beta H') \end{aligned}$$

in terms of which the Higgs equation becomes

$$(b^2 D)^\cdot = \frac{1}{r^2} \left[ b^2 r^2 \left( \frac{\alpha}{a} C + \beta D \right) \right]' - \frac{2H\alpha a}{r^2} (w^2 + \tilde{w}^2) - \lambda H \alpha a b^2 (H^2 - \eta^2)$$

The consistency requirement  $(\dot{H})^\cdot = (H')^\cdot$  leads to the equation

$$\dot{C} = \left( \frac{\alpha}{a} D + \beta C \right)^\cdot.$$

## Special Cases

Let's now consider these equations in the limit of flat space. In this case, of course  $\alpha = a = b = 1$  and  $\beta = 0$ . We have also to make a gauge choice for the Yang-Mills field. We choose  $v = 0$ . In this case

$$\begin{aligned}\Pi &= \dot{w} - u\tilde{w} \\ \Phi &= w' \\ P &= \dot{\tilde{w}} + uw \\ Q &= \tilde{w}'\end{aligned}$$

The YMH equations now become

$$\begin{aligned}-(\dot{w} - u\tilde{w})' + w'' + u(\dot{\tilde{w}} + uw) + \frac{w}{r^2}(1 - w^2 - \tilde{w}^2) &= g^2 H^2 w \\ -(\dot{\tilde{w}} + uw)' + \tilde{w}'' - u(\dot{w} - u\tilde{w}) + \frac{\tilde{w}}{r^2}(1 - w^2 - \tilde{w}^2) &= g^2 H^2 \tilde{w} \\ -\left[\frac{r^2}{2}u'\right]' &= \tilde{w}(\dot{w} - u\tilde{w}) - w(\dot{\tilde{w}} + uw) \\ -\left[\frac{r^2}{2}u'\right]' &= \tilde{w}w' - w\tilde{w}'.\end{aligned}$$

Note that in the static case, the fourth equation implies  $\tilde{w} = Cw$  and as mentioned before a constant gauge transformation is all that is needed to set  $C$  to zero. One can then simplify the equations even further and reduce the equations to those for the static dyon solution of Julia and Zee. By setting  $u = 0$  in those equations one gets the equations for the t'Hooft-Polyakov monopole (see for example, Rajaraman p71).

If we consider, however, the time dependant case (even in flat space!), I am no longer convinced that Zhou's no dyon argument holds. Zhou is arguing for  $u = 0$  in the absence of the Higgs field. However, he starts off assuming that  $\tilde{w} = 0$ . As we said before in the static case,  $\tilde{w}$  can be considered to be pure gauge. But I don't see that we can say that in the non-static case. If indeed  $\tilde{w} = 0$  a priori, then  $u'$  is time independent and Zhou's argument goes through to the point that  $u = 0$ , but in general, I don't see why  $\tilde{w}$  should be zero. In behalf of Zhou, there are several non-existence theorems to the effect that in flat space and without other matter, there are no everywhere regular solutions of the Yang-Mills equations. However, I don't know of any such theorems for the time dependant case (although I can't say I have looked very hard nor do I know the literature on time-dependant solutions to the Yang-Mills equations). So it seems there may well be dynamical configurations which have some sort of charge. The crux of this, though, is that I think we are safer keeping both  $u$  and  $\tilde{w}$  until we either come up with a convincing argument to set them to zero. They may indeed turn out to be pure gauge, but at this point, I don't see how. It is definitely something to think about further.

Another limit to consider is radial gauge ( $\beta = 0$  and  $b = 1$ ) together with the static case. Again, make the choice of gauge  $v = 0$ . The auxiliary variables then become

$$\begin{aligned}\Pi &= -\frac{a}{\alpha}u\tilde{w} \\ \Phi &= w' \\ P &= \frac{a}{\alpha}uw \\ Q &= \tilde{w}'\end{aligned}$$

As a result of assuming time independance we have the exact same argument as in the last paragraph and we can again set  $\tilde{w} = 0$ . We now have  $Q = \Pi = B = 0$  and  $P = \frac{a}{\alpha}uw$  and  $\phi = w'$ . The equations reduce



after some algebra to

$$\begin{aligned}
\left(\frac{\alpha}{a}w'\right)' &= -\frac{a}{\alpha}wu^2 - \frac{\alpha a}{r^2}w(1-w^2) + g^2\alpha awH^2 \\
\left(\frac{r^2}{\alpha a}u'\right)' &= \frac{2a}{\alpha}uw^2 \\
\left(\frac{\alpha}{a}r^2H'\right)' &= 2\alpha aHw^2 + \lambda\alpha ar^2H(H^2 - \eta^2) \\
\frac{a'}{a} &= \frac{1-a^2}{2r} + 4\pi Gra^2\rho \\
\frac{\alpha'}{\alpha} &= \frac{a^2-1}{2r} + 4\pi Gra^2S^r_r
\end{aligned}$$

where  $\rho$  and  $S^r_r$  are given in the final section. These equations reduce to those of Breitenlohner *et al* for the purely magnetic monopole case  $u = 0$  and to the equations of Bartnik and McKinnon for the no Higgs case  $u = H = \lambda = 0$ . We also recover the prior equations for the flat space dyon or magnetic monopole when  $a = \alpha = 1$ .

One final case to consider is that considered by Choptuik *et al* as well as Straumann and Zhou and others. This is the *time dependent* case in radial gauge with no Higgs field ( $H = \lambda = 0$ ) and the assumption of  $u = v = \tilde{w} = 0$ . We then have  $K^\theta_\theta = \beta = P = Q = 0$  and  $b = 1$ . Using the auxiliary variables now, the equations are

$$\begin{aligned}
\dot{\Pi} &= \left(\frac{\alpha}{a}\Phi\right)' + \frac{\alpha a}{r^2}w(1-w^2) \\
\dot{\Phi} &= \left(\frac{\alpha}{a}\Pi\right)' \\
\dot{a} &= \frac{4\pi G}{g^2} \frac{\alpha}{r} \Pi \Phi \\
\frac{a'}{a} &= \frac{1-a^2}{2r} + 4\pi Gra^2\rho \\
\frac{\alpha'}{\alpha} &= \frac{a^2-1}{2r} + 4\pi Gra^2S^r_r
\end{aligned}$$

where

$$\begin{aligned}
\rho &= \frac{1}{2g^2r^2a^2} \left[ \Phi^2 + \Pi^2 + \frac{a^2}{2r^2}(1-w^2)^2 \right] \\
S^r_r &= \frac{1}{2g^2r^2a^2} \left[ \Phi^2 + \Pi^2 - \frac{a^2}{2r^2}(1-w^2)^2 \right]
\end{aligned}$$

**Altogether now ...**

Okay, so let's summarize all this in a hopefully cogent way. We start with the gauge connection

$$A = u\tau_r dt + v\tau_r dr + (w\tau_\theta + \tilde{w}\tau_\phi)d\theta + (\cot\theta\tau_r + w\tau_\phi - \tilde{w}\tau_\theta)\sin\theta d\phi.$$

where  $[\tau_i, \tau_j] = \epsilon_{ijk}\tau_k$  with  $i, j, k \in \{r, \theta, \phi\}$ . This is invariant under a gauge transformation of the form  $U = e^{\psi(t,r)\tau_r}$ . The field strength derived from this connection is

$$\begin{aligned} F &= \tau_r(\dot{v} - u')dt \wedge dr \\ &+ [(\dot{w} - u\tilde{w})dt + (w' - v\tilde{w})dr] \wedge (\tau_\theta d\theta + \tau_\phi \sin\theta d\phi) \\ &+ [(\dot{\tilde{w}} + uw)dt + (\tilde{w}' + vw)dr] \wedge (\tau_\phi d\theta - \tau_\theta \sin\theta d\phi) \\ &- (1 - w^2 - \tilde{w}^2)\tau_r d\theta \wedge \sin\theta d\phi \end{aligned}$$

The full Lagrangian for EYMH is

$$\mathcal{L} = \sqrt{-g} \left[ \frac{R}{16\pi G} - \frac{1}{g^2} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} D_\mu \Phi^a D^\mu \Phi^a - V(\Phi) \right].$$

where

$$V(\Phi^a) = \frac{\lambda}{4} (\Phi^a \Phi^a - \eta^2)^2$$

The equations of motion are found by varying the action with respect to the fields. Varying with respect to the metric gives

$$\begin{aligned} \frac{1}{16\pi G} G_{\mu\nu} &= T_{\mu\nu} \\ &= \frac{1}{g^2} (2F_{\mu\lambda}^a F^{a\nu\lambda} - \frac{1}{2} g_{\mu\nu} F_{\alpha\beta}^a F^{a\alpha\beta}) \\ &+ \frac{1}{2} (D_\mu \Phi^a D_\nu \Phi^a - \frac{1}{2} g_{\mu\nu} (D_\lambda \Phi^a D^\lambda \Phi^a)) - \frac{1}{2} g_{\mu\nu} V(\Phi^a). \end{aligned}$$

Varying with respect to the connection  $A_\mu^a$  yields

$$D_\mu F^{a\mu\nu} = \frac{g^2}{2} \epsilon^{abc} \Phi^b D^\nu \Phi^c$$

or in a form which reveals its general covariance

$$\nabla_\mu F^{a\mu\nu} + \epsilon^{abc} A_\mu^b F^{c\mu\nu} = g^2 \epsilon^{abc} \Phi^b [\nabla^\nu \Phi^c + \epsilon^{cde} A^{d\nu} \Phi^e]$$

Finally, varying with respect to the Higgs fields gives us

$$\begin{aligned} D^\mu D_\mu \Phi^a &= \frac{\partial V}{\partial \Phi^a} \\ &= \lambda \Phi^a (\Phi^b \Phi^b - \eta^2) \end{aligned}$$

or again in a generally covariant form

$$g^{\mu\nu} [\nabla_\mu \nabla_\nu \Phi^a + \epsilon^{abc} A_\mu^b \nabla_\nu \Phi^c + \epsilon^{abc} \nabla_\mu (A_\nu^b \Phi^c) + \epsilon^{ade} A_\mu^d (\epsilon^{ebc} A_\nu^b \Phi^c)] = \lambda \Phi^a (\Phi^b \Phi^b - \eta^2)$$

The metric with which we work is

$$ds^2 = (-\alpha^2 + a^2 \beta^2) dt^2 + 2a^2 \beta dt dr + a^2 dr^2 + r^2 b^2 d\Omega^2$$

Working in the reverse order of the equations we have presented, with the ansatz of

$$\Phi^a = \delta_r^a H(t, r)$$

for the Higgs field, the only non-trivial component of the Higgs equation is

$$\nabla_\mu \nabla^\mu H = \frac{2H}{r^2 b^2} (w^2 + \tilde{w}^2) + \lambda H (H^2 - \eta^2).$$

We write this as

$$(b^2 D) \cdot = \frac{1}{r^2} \left[ b^2 r^2 \left( \frac{\alpha}{a} A + \beta D \right) \right]' - \frac{2H\alpha a}{r^2} (w^2 + \tilde{w}^2) - \lambda H \alpha a b^2 (H^2 - \eta^2)$$

together with

$$\dot{C} = \left( \frac{\alpha}{a} D + \beta C \right)'$$

where we have used the variables

$$\begin{aligned} C &= H' \\ D &= \frac{a}{\alpha} (\dot{H} - \beta H'). \end{aligned}$$

The equations of motion for the Yang-Mills fields are now

$$\begin{aligned} -\dot{\Pi} + \left[ \beta \Pi + \frac{\alpha}{a} \Phi \right]' + uP - v \left( \beta P + \frac{\alpha}{a} Q \right) + \frac{\alpha a}{b^2 r^2} w (1 - w^2 - \tilde{w}^2) &= g^2 \alpha a w H^2 \\ -\dot{P} + \left[ \beta P + \frac{\alpha}{a} Q \right]' - u\Pi + v \left( \beta \Pi + \frac{\alpha}{a} \Phi \right) + \frac{\alpha a}{b^2 r^2} \tilde{w} (1 - w^2 - \tilde{w}^2) &= g^2 \alpha a \tilde{w} H^2 \end{aligned}$$

$$Y' = \tilde{w} \Pi - w P$$

$$\dot{Y} = \frac{\alpha}{a} (\tilde{w} \Phi - w Q) + \beta (\tilde{w} \Pi - w P)$$

with the definitions

$$\begin{aligned} \Pi &= \frac{a}{\alpha} [\dot{w} - u\tilde{w} - \beta(w' - v\tilde{w})] \\ \Phi &= w' - v\tilde{w} \\ P &= \frac{a}{\alpha} [\dot{\tilde{w}} + uw - \beta(\tilde{w}' + vw)] \\ Q &= \tilde{w}' + vw \\ Y &= \frac{b^2 r^2}{2\alpha a} (\dot{v} - u') \end{aligned}$$

and the consistency equations

$$\begin{aligned} \dot{\Phi} &= \left[ \frac{\alpha}{a} \Pi + \beta \Phi \right]' + uQ - v \left[ \frac{\alpha}{a} P + \beta Q \right] - \tilde{w} \frac{2\alpha a}{b^2 r^2} Y \\ \dot{Q} &= \left[ \frac{\alpha}{a} P + \beta Q \right]' - u\Phi + v \left[ \frac{\alpha}{a} \Pi + \beta \Phi \right] + w \frac{2\alpha a}{b^2 r^2} Y. \end{aligned}$$

Finally, the relevant Einstein equations are the evolution equations for the metric components and the components of the extrinsic curvature together with the Hamiltonian and momentum constraints as follows:

$$\begin{aligned} \dot{a} &= -a\alpha K^r_r + (a\beta)' \\ \dot{b} &= -\alpha b K^\theta_\theta + \frac{\beta}{r} (rb)' \end{aligned}$$

$$\begin{aligned}
\dot{K}^r_r &= \beta K^r_r + \alpha K^r_r K - \frac{1}{a} \left( \frac{\alpha'}{a} \right)' - \frac{2\alpha}{arb} \left[ \frac{(rb)'}{a} \right]' + 4\pi G\alpha [S - \rho - 2S^r_r] \\
\dot{K}^\theta_\theta &= \beta K^\theta_\theta + \alpha K^\theta_\theta K + \frac{\alpha}{(rb)^2} - \frac{1}{a(rb)^2} \left( \frac{\alpha rb}{a} (rb)' \right)' + 4\pi G\alpha [S - \rho - 2S^\theta_\theta] \\
&\quad - \frac{2}{arb} \left[ \left( \frac{(rb)'}{a} \right)' + \frac{1}{rb} \left( \left( \frac{rb}{a} (rb)' \right)' - a \right) \right] + 4K^r_r K^\theta_\theta + 2K^{\theta 2}_\theta = 16\pi G\rho \\
&\quad - \frac{(rb)'}{rb} (K^\theta_\theta - K^r_r) - K^{\theta'}_\theta = 4\pi G j_r
\end{aligned}$$

After some calculation (okay, lots of it), I find the following useful quantities:

$$F^a_{\mu\nu} F^{a\mu\nu} = -\frac{2Y^2}{b^4 r^4} + \frac{(1 - w^2 - \tilde{w}^2)^2}{2b^4 r^4} + \frac{1}{b^2 r^2 a^2} [Q^2 + \Phi^2 - P^2 - \Pi^2]$$

$$D_\mu \Phi^a D^\mu \Phi^a = \frac{1}{a^2} (C^2 - D^2) + \frac{2H^2}{b^2 r^2} (w^2 + \tilde{w}^2)$$

$$\begin{aligned}
\rho &= \frac{1}{4g^2} \left\{ \frac{4Y^2}{b^4 r^4} + \frac{(1 - w^2 - \tilde{w}^2)^2}{b^4 r^4} + \frac{2}{b^2 r^2 a^2} [Q^2 + \Phi^2 + P^2 + \Pi^2] \right\} \\
&\quad + \frac{1}{4} \left\{ \frac{1}{a^2} (C^2 + D^2) + \frac{2H^2}{b^2 r^2} (w^2 + \tilde{w}^2) \right\} + \frac{1}{2} V(\Phi^a)
\end{aligned}$$

$$\begin{aligned}
S^r_r &= \frac{1}{4g^2} \left\{ -\frac{4Y^2}{b^4 r^4} - \frac{(1 - w^2 - \tilde{w}^2)^2}{b^4 r^4} + \frac{2}{b^2 r^2 a^2} [Q^2 + \Phi^2 + P^2 + \Pi^2] \right\} \\
&\quad + \frac{1}{4} \left\{ \frac{1}{a^2} (C^2 + D^2) - \frac{2H^2}{b^2 r^2} (w^2 + \tilde{w}^2) \right\} - \frac{1}{2} V(\Phi^a)
\end{aligned}$$

$$S^\theta_\theta = \frac{1}{4g^2} \left\{ \frac{4Y^2}{b^4 r^4} + \frac{(1 - w^2 - \tilde{w}^2)^2}{b^4 r^4} \right\} + \frac{1}{4} \frac{1}{a^2} (-C^2 + D^2) - \frac{1}{2} V(\Phi^a)$$

$$S^\phi_\phi = S^\theta_\theta$$

$$j_r = -\frac{1}{g^2 a b^2 r^2} (\Pi\Phi + PQ) - \frac{1}{2a} (CD)$$

### The problem at hand

We are now in a position to solve these equations in a particular coordinate system and with a particular choice of gauge. Although, we gave some special cases before, we will use this section to write out our equations as we want to solve them. We choose maximal slicing (the trace of the extrinsic curvature,  $K$ , vanishes) and radial coordinates,  $b = 1$ . For the gauge, we pick  $v = 0$ . This does not completely specify the gauge since a gauge transformation of the form  $U = e^{\psi(t)\tau_r}$  would leave the connection form invariant. We will fix this later when we discuss boundary conditions on the Yang-Mills fields.

Given these conditions, the evolution equations are

$$\begin{aligned}
\dot{\Pi} &= \left[ \beta\Pi + \frac{\alpha}{a}\Phi \right]' + uP + \frac{\alpha a}{r^2}w(1 - w^2 - \tilde{w}^2) - g^2\alpha awH^2 \\
\dot{P} &= \left[ \beta P + \frac{\alpha}{a}Q \right]' - u\Pi + \frac{\alpha a}{r^2}\tilde{w}(1 - w^2 - \tilde{w}^2) - g^2\alpha a\tilde{w}H^2 \\
\dot{\Phi} &= \left[ \frac{\alpha}{a}\Pi + \beta\Phi \right]' + uQ - \tilde{w}\frac{2\alpha a}{r^2}Y \\
\dot{Q} &= \left[ \frac{\alpha}{a}P + \beta Q \right]' - u\Phi + w\frac{2\alpha a}{r^2}Y \\
\dot{C} &= \left( \frac{\alpha}{a}D + \beta C \right)' \\
\dot{D} &= \frac{1}{r^2} \left[ r^2 \left( \frac{\alpha}{a}C + \beta D \right) \right]' - \frac{2H\alpha a}{r^2}(w^2 + \tilde{w}^2) - \lambda H\alpha a(H^2 - \eta^2) \\
\dot{w} &= \frac{\alpha}{a}\Pi + u\tilde{w} + \beta w' \\
\dot{\tilde{w}} &= \frac{\alpha}{a}P - uw + \beta\tilde{w}' \\
\dot{H} &= \frac{\alpha}{a}D + \beta C \\
\dot{Y} &= \frac{\alpha}{a}(\tilde{w}\Phi - wQ) + \beta(\tilde{w}\Pi - wP)
\end{aligned}$$

while the constraints are

$$\begin{aligned}
w' &= \Phi \\
\tilde{w}' &= Q \\
u' &= -\frac{2\alpha a}{r^2}Y \\
Y' &= \tilde{w}\Pi - wP \\
\alpha'' &= \alpha' \left( \frac{a'}{a} - \frac{2}{r} \right) + \frac{2\alpha}{r^2} \left( a^2 - 1 + \frac{2ra'}{a} \right) + 4\pi G\alpha(S - 3\rho) \\
a' &= a\frac{1 - a^2}{2r} + \frac{3}{2}ra^3K_\theta^{\theta 2} + 4\pi Gra\rho \\
K_\theta^{\theta'} &= -\frac{3}{r}K_\theta^\theta + 4\pi G \left\{ \frac{1}{g^2ar^2}(\Pi\Phi + PQ) + \frac{CD}{2a} \right\}
\end{aligned}$$

where

$$\begin{aligned}
S - 3\rho &= \frac{2Y^2}{g^2r^4} + \frac{(1 - w^2 - \tilde{w}^2)^2}{2g^2r^4} + \frac{1}{g^2r^2a^2}(Q^2 + \Phi^2 + P^2 + \Pi^2) + \frac{C^2}{a^2} \\
&\quad + \frac{2H^2}{r^2}(w^2 + \tilde{w}^2) + \frac{3\lambda}{4}(H^2 - \eta^2) \\
\rho &= \frac{Y^2}{g^2r^4} + \frac{(1 - w^2 - \tilde{w}^2)^2}{4g^2r^4} + \frac{1}{2g^2r^2a^2}(Q^2 + \Phi^2 + P^2 + \Pi^2) + \frac{1}{4a^2}(C^2 + D^2) \\
&\quad + \frac{H^2}{2r^2}(w^2 + \tilde{w}^2) + \frac{\lambda}{8}(H^2 - \eta^2)^2
\end{aligned}$$

and we have the algebraic relation

$$\beta = \alpha r K_\theta^\theta.$$

Now we need to consider the boundary conditions for these equations. As is usual, we demand regularity at the origin,  $r = 0$ . We can deduce from the  $\alpha$  and  $a$  equations that  $\alpha'(t, 0) = a'(t, 0) = 0$  as well as  $a(t, 0) = 1$ . In addition,  $\beta(t, 0) = K_\theta^\theta(t, 0) = 0$ .

As for the matter fields, a little analysis reveals that we must satisfy either

$$w(t, 0)^2 + \tilde{w}(t, 0)^2 = 1 \quad \text{and} \quad H(t, 0) = 0$$

or

$$w(t, 0) = \tilde{w}(t, 0) = 0 \quad \text{and} \quad H'(t, 0) = 0$$

The second case forces the Yang-Mills fields to be identically zero (*i.e.* all the derivatives are zero as well). So if we want to have in general a nontrivial Yang-Mills field and at the same time include a Higgs field, our boundary conditions must consist of the first case.

The curiosity, perhaps, is that nothing in the equations fixes both  $\dot{w}(t, 0)$  and  $\dot{\tilde{w}}(t, 0)$ . We may parameterize them as

$$w(t, 0) = \cos \xi(t) \quad \text{and} \quad \tilde{w}(t, 0) = \sin \xi(t)$$

where  $\xi(t)$  is an arbitrary function. Some more work shows that with this parameterization, we have the additional boundary conditions:

$$u(t, 0) = -\dot{\xi}(t, 0), \quad w'(t, 0) = k \sin \xi(t), \quad \text{and} \quad \tilde{w}'(t, 0) = -k \cos \xi(t)$$

The solution, of course, is that we still have residual gauge freedom under which  $u(t, r) \rightarrow u(t, r) + \dot{\psi}(t)$ . We now fix some of this additional gauge freedom by setting  $\dot{\psi} = -\dot{\xi}$ . This is equivalent to choosing  $u(t, 0) = 0$  or letting  $\xi$  be a constant. There is still the freedom to perform a constant gauge transformation  $U = e^{\psi\tau r}$  where  $\psi$  is now a constant. But this has the effect of sending  $w$  and  $\tilde{w}$  into linear combinations of each other. So we fix this last bit of gauge freedom by setting  $\xi = 0$ .

With these choices of gauge, we now have the following boundary conditions:  $w(t, 0) = 1$ ,  $\tilde{w}(t, 0) = u(t, 0) = 0$ ,  $w'(t, 0) = 0$ , and  $\tilde{w}'(t, 0) = -k$ . The constant,  $k$ , can be shown to be zero from regularity or if we demand that the energy density is finite. This results then in the following quantities being zero at  $r = 0$ :  $\Pi, \Phi, P, Q$ , and  $Y$ .

At the outer boundaries, we impose an outgoing condition. This simply assumes that there is no radiation coming in from outside our mesh. This is not completely true as in general there will be backscattering of the propagating fields off regions of high curvature, but if our domain of integration is large enough, the contributions from this scattering would hopefully be negligible.

We will show the method for a representative field  $\psi(t, r)$  which has an outgoing form together with some fall-off given by

$$\psi(t, r) \sim \psi_\infty + \frac{f(r - v_c t)}{r^p}$$

where the characteristic speed is  $v_c = \pm\alpha/a - \beta$  and we have allowed for the possibility that the field takes a constant nonzero value at infinity  $\psi_\infty$  (such as the Higgs field). Taking the positive sign for outgoing radiation, we substitute into the relation

$$\dot{f} + v_c f' = 0$$

and rearrange to get

$$\dot{\psi} + \left(\frac{\alpha}{a} - \beta\right) \left(\psi' + \frac{p}{r}(\psi - \psi_\infty)\right) = 0$$

If we define the usual sort of variables:

$$\Phi = \psi' \quad \text{and} \quad \Pi = \frac{a}{\alpha} \left(\dot{\psi} - \beta\psi'\right)$$

we can rewrite this as

$$\Phi + \Pi + \left(1 - \beta\frac{a}{\alpha}\right) \frac{p}{r}(\psi - \psi_\infty) = 0$$

Taking an extra spatial derivative allows us to write an outgoing condition for  $\Phi$ :

$$\dot{\Phi} + \left[ \left( \frac{\alpha}{a} - \beta \right) \left( \Phi + \frac{p}{r} (\psi - \psi_\infty) \right) \right]' = 0$$

In terms of the radiative variables we have been using, for the Higgs field  $H$ ,  $C$  and  $D$  take the place of the above  $\Phi$  and  $\Pi$  respectively with  $p = 1$ . For the Yang-Mills potential,  $w$ , our previous  $\Phi$  and  $\Pi$  obey the above equations with  $p = 0$  and for  $\tilde{w}$ ,  $Q$  and  $P$  obey the above equations for  $\Phi$  and  $\Pi$  respectively with  $p = 0$ .<sup>11</sup>

### Another problem

We would also like to consider the generalization of Choptiuk *et al* namely by moving away from the magnetic ansatz and considering a more general spherically symmetric Yang-Mills system. For this problem then, we set the Higgs field to zero and consider the Einstein equations in polar/radial coordinates ( $b = 1$ ,  $K_\theta^\theta = 0$ , and  $\beta = 0$ ). The equations are very similar to before with, albeit, some important modifications.

Given these conditions, the evolution equations are

$$\begin{aligned} \dot{\Pi} &= \left[ \frac{\alpha}{a} \Phi \right]' + uP + \frac{\alpha a}{r^2} w(1 - w^2 - \tilde{w}^2) \\ \dot{P} &= \left[ \frac{\alpha}{a} Q \right]' - u\Pi + \frac{\alpha a}{r^2} \tilde{w}(1 - w^2 - \tilde{w}^2) \\ \dot{\Phi} &= \left[ \frac{\alpha}{a} \Pi \right]' + uQ - \tilde{w} \frac{2\alpha a}{r^2} Y \\ \dot{Q} &= \left[ \frac{\alpha}{a} P \right]' - u\Phi + w \frac{2\alpha a}{r^2} Y \\ \dot{w} &= \frac{\alpha}{a} \Pi + u\tilde{w} \\ \dot{\tilde{w}} &= \frac{\alpha}{a} P - uw \\ \dot{Y} &= \frac{\alpha}{a} (\tilde{w}\Phi - wQ) \end{aligned}$$

while the constraints are

$$\begin{aligned} w' &= \Phi \\ \tilde{w}' &= Q \\ u' &= -\frac{2\alpha a}{r^2} Y \\ Y' &= \tilde{w}\Pi - wP \\ \frac{\alpha'}{\alpha} &= \frac{a^2 - 1}{2r} + 4\pi G r a^2 S_r{}^r \\ \frac{a'}{a} &= \frac{1 - a^2}{2r} + 4\pi G r a^2 \rho \end{aligned}$$

where

$$\begin{aligned} S_r{}^r &= -\frac{Y^2}{g^2 r^4} - \frac{(1 - w^2 - \tilde{w}^2)^2}{4g^2 r^4} + \frac{1}{2g^2 r^2 a^2} (Q^2 + \Phi^2 + P^2 + \Pi^2) \\ \rho &= \frac{Y^2}{g^2 r^4} + \frac{(1 - w^2 - \tilde{w}^2)^2}{4g^2 r^4} + \frac{1}{2g^2 r^2 a^2} (Q^2 + \Phi^2 + P^2 + \Pi^2) \end{aligned}$$

We can also include the evolution equation for  $a$

$$\begin{aligned} \dot{a} &= -\alpha a K_r{}^r \\ &= \frac{4\pi G \alpha}{g^2 r} (\Pi\Phi + PQ). \end{aligned}$$

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<sup>11</sup> We note that in the far field regime,  $w^2 + \tilde{w}^2$  can be 0 or 1 depending on the nature of the solution. In the former case the field  $u(t, r)$  should fall off as  $1/r$  while for the case that the fields go to 1, the falloff of  $u(t, r)$  should be  $1/r^2$ . At least I think so ...

The boundary conditions for these equations are the same as before. As usual, we demand regularity at the origin,  $r = 0$ . We can deduce from the  $\alpha$  and  $a$  equations that  $\alpha'(t, 0) = a'(t, 0) = 0$  as well as  $a(t, 0) = 1$ . For the matter fields, we choose (gauge choice and consequences of regularity and finite energy density)  $w(t, 0) = 1$ ,  $\tilde{w}(t, 0) = u(t, 0) = 0$ ,  $w'(t, 0) = 0$ , and  $\tilde{w}'(t, 0) = 0$ . This results then in the following quantities being zero at  $r = 0$ :  $\Pi, \Phi, P, Q$ , and  $Y$ . We also impose outgoing boundary conditions as described before.



## Bibliography

This is by no means a complete bibliography, but it is a list of those references that confused me less than most. The only order to these references is the way they now happen to be stacked in my folder. Also, be aware that conventions seem to change according to a Weiner process – randomly.

J. Harvey, "Magnetic Monopoles, Duality, and Supersymmetry," hep-th/9603086.

Despite the title, this is a relatively good introduction to the flat space Yang-Mills-Higgs system and to some of the more "modern" conventions. He does use the high energy signature for the metric of (1,-1,-1,-1) and I do quibble over the overall sign of his eq 1.23, but at least he writes the equations down.

Rajaraman, "Solitons and Instantons."

A good book. It again writes down the equations as well as the specific equations for the t'Hooft-Polyakov monopole and the Julia-Zee dyon. Important check for me.

Julia and Zee, "Poles with both magnetic and electric charges in non-Abelian gauge theory," Phys Rev D 11 # 8, 2227 (1975).

Paper generalizing t'Hooft-Polyakov monopole to dyons. They write down equations and their ansatz for solving them. Okay.

Ortiz, Miguel "Curved-space magnetic monopoles," Phys Rev D 45 #8, R2586 (1992).

He solves the static EYM equations for monopoles. He does not address stability issues.

Tachizawa, Maeda, and Torii "Non-Abelian black holes and catastrophe theory," Phys Rev D 51 #8, 4054 (1995).

Gives the static equations (ODEs) for EYM (magnetic monopoles).

Greene, Mathur and O'Neill, "Eluding the no-hair conjecture: Black holes in spontaneously broken gauge theories," Phys Rev D 47 #6, 2242 (1993).

Again they give the static equations, but for complex scalar field in the fundamental representation. They do have a discussion, however, of the nature of the ansatz for the gauge connection and seem to argue that  $\tilde{w} \neq 0$  in general. This is something I need to spend some more time with.

Breitenlohner, Forgacs, and Maison "Gravitating monopole solutions," Nuc Phys B 383 (1992) 357.

They discuss monopole solutions of EYM equations. They give static equations. Fairly complete discussion of their results. They do not discuss stability, but it is discussed in Phys Lett B 338 (1994) 181 by Helia Holliman (?) or some such. He/she finds that there are solutions that are stable in linear perturbation theory. Interesting result since many of these objects in EYM (BM particle-like solutions, non-abelian black holes, etc) seem to be unstable.

Choptuik, Chmaj, Bizon "Critical Behaviour in Gravitational Collapse of a Yang-Mills Field," gr-qc/9603051.

Matt's work. My jumping off point.

Zhou, "Instability of  $SU(2)$  Einstein-Yang-Mills Solitons and Non-Abelian Black Holes," Helv. Phys. Acta 65 (1992).

This is Zhou's dissertation essentially. I started with it and then had to go elsewhere to figure out some of his conventions. Still, not too bad.

Ershov and Galtsov, Phys Lett A150 (1990) 159.

I don't remember the title, but they were making some proof of no non-abelian hair I think. They discuss the gauge connection and mention the fact most people gloss over of using a spherical projection of the Pauli spin matrices.

Xanthopoulos "Classical Aspects of Yang-Mills Theories," in "Solutions of Einstein's Equations: Techniques and Results." 1984.

A nice introduction for relativists to classical Yang-Mills theories. With Harvey, the first references that I found that explained that many people use anti-Hermitian generators for the basis of  $SU(2)$  – originally news to me.

Witten, "Some Exact Multipseudoparticle Solutions of Classical Yang-Mills Theory," Phys Rev Lett 38 #3, 121 (1977).

The original source for the  $SU(2)$  gauge ansatz. I didn't use it for much more than that.

Bartnik and McKinnon,

I can't find the reference now, but of course the original discovery of the particle like solutions to EYM equations.

Forgacs and Manton, "Space-Time Symmetries in Gauge Theories," *Comm in Math Phys.* 72, 15 (1980).

Wasn't that helpful, but did try to derive and explain Witten's ansatz for spherically symmetric  $SU(2)$ . They work of course in flat space.