QUANTUM MECHANICS IN SNYDER SPACE

by

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ABSTRACT

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I present a systematic study of Snyder space, the original quantized spacetime described by Hartland Snyder in Phys. Rev. **71**, 38-41. I outline characteristics of representations in both an underlying de Sitter space and momentum space and discuss how information about position can be recovered. I present methods for studying systems in Snyder space, which I use to find the energy spectrum of the harmonic oscillator in one and two dimensions. I discuss the relation between Snyder space and noncommutative quantum mechanics and its place in theories of unification and quantum gravity. Algorithms that I developed for the manipulation of noncommutative objects and for the evaluation of formulas from perturbation theory are included.

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Chapter 1

Introduction

1.1 Motivation

In this thesis I discuss the properties of Snyder space, which is a formulation of quantum mechanics on a quantized spacetime. Snyder space holds an important historical place since it is the first formulation of quantum mechanics on a noncommutative manifold. There has been much work recently studying consequences of formulating quantum mechanics on noncommutative manifolds, and it has become commonplace to cite Snyder space for its historical precedence. However, very little has actually been done to study the properties of Snyder space. In this work I seek to alleviate this problem by presenting a systematic overview of Snyder space and compare its results to similar results found in the literature for other noncommutative spaces.

The question I am addressing is open-ended. I have been influenced to a small extent by the results that can be found in the literature for other noncommutative spaces. Partially for this reason I chose to study the simple harmonic oscillator in Snyder space, which has proven to be a fruitful subject in other spaces. However, this research has primarily been an exploratory process; I did not know a priori what the

results would be, just as Lewis and Clark did not know exactly what they would find on their journey to the Pacific Ocean. This exploration is characteristic of theoretical physics in general. At present there is no significant discrepancy between theory and experimental results in high-energy physics. In spite of this agreement, there is some dissatisfaction with certain aspects of the theory. In particular, the two major theories of theoretical physics, quantum field theory and general relativity, are apparently incompatible. A major goal is to unify these theories into a single unified theory. As such, theoretical physicists make conjectures about possible alterations that could be made to a theory that could lead to unification, and then catalog results that experimentalists can seek to verify or to annul. Some proposed alterations are taken more seriously than others, and some conjectures are considered by mathematical physicists solely as mathematical exercises.

I chose to study Snyder space for several reasons. First, as I have already mentioned, it holds a significant place in the history of physics as being the first formulation of quantum mechanics on a noncommutative manifold, but it has not been seriously studied since it original conception in the 1940s. Secondly, Snyder space was originally proposed to solve a problem in quantum field theory, and while it was never shown to be deficient in solving this problem, it has become obvious that it is closely related to proposed theories of quantum gravity far removed from its original context. Also, the commutation algebra of Snyder space has been derived by methods independent of Snyder's original derivation, suggesting that it could be a viable model of nature. Finally, a formulation of quantized spacetime is an interesting exercise in mathematical physics, the results of which should be cataloged and investigated.

In this first chapter, I elaborate on each of these reasons for reconsidering Snyder space and outline the structure of the thesis as a whole.

1.2 Historical significance of Snyder space

Snyder space is a formulation of quantum mechanics on a quantized spacetime that Hartland S. Snyder proposed in 1947 [1]. The significance of this work has only been realized recently; however, Snyder's other work has long been appreciated for its theoretical importance. For more information about Hartland Snyder and his contributions to physics, see appendix A.

With his proposition of quantized spacetime, Snyder's aim was to solve "[t]he problem of the interaction of matter and fields." In his theory, he redefines the fundamental operators of quantum mechanics in such a way that the position operators take on discrete values, thereby introducing a fundamental length. Snyder hopes that "the introduction of such a unit of length will remove many of the divergence troubles of the present field theory."

Within a year, he published a second paper in which he outlined some aspects of quantum field theory in the framework of his previous paper [2]. This however is the last paper that he published on the subject, and he moved on to other matters. The problem of the infinities in quantum field theory found a solution with the development of renormalization theory in the late forties, and Snyder's work was largely forgotten.

Others had considered the idea of a discrete spacetime before Snyder. Heisenberg considered doing physics on a spacetime lattice over a decade before Snyder [3]. However, Snyder was the first to formally present the details of quantum mechanics on a quantized space. In the decades that followed Snyder's work, others attempted to describe space and time as a discrete space, see for example [4], but their motivation was not to fix fundamental flaws in any theory. Instead, their aim was to survey the consequences of this assumption. So it appeared that Snyder's quantized

spacetime would be destined to be of little, if any, long term consequence. That, however, changed in the 1980s as work on theories of quantum gravity suggested that spacetime may actually be quantized.

1.3 Reasons for reconsidering Snyder space

Snyder space has recently found renewed interest because of its connection with theories of quantum gravity and a unified theory of physics. The search for a unified theory of physics predates even Snyder space, although the unification of gravity and quantum theory is a relatively recent pursuit. Modern candidates for a theory of quantum gravity include string theory and loop quantum gravity. Loop quantum gravity was the first major theory to postulate that spacetime must be quantized [5]. String theory eventually came to the same conclusion, and so among these theories it is generally accepted that at some scale, spacetime must be discrete. There is no general consensus, however, at what length these effects will be observed [6].

While there is agreement that spacetime is quantized, there is disagreement as to how the quantization manifests itself. Snyder's method of quantization has the appealing characteristic that it is Lorentz invariant, meaning it is invariant under boosts and rotations. Most models of quantized spacetime are not Lorentz invariant. In fact, before Snyder's work, it was generally believed that Lorentz invariance required that position have a continuous spectrum [3].

Another appealing aspect of Snyder space is that Snyder derived his model from first principles. His goal was to find a Lorentz invariant spacetime with a discrete spectrum that modified as little as possible the other properties of quantum mechanics. Thus, other aspects of quantum mechanics, such as the nature of angular momentum, remain essentially unchanged in Snyder space.

In addition, there are mathematical reasons for finding Snyder space a promising subject of investigation. These reasons originate in the commutation algebra that Snyder space predicts. The following sections explain the significance of Snyder's commutation algebra.

1.4 Noncommutative objects in theoretical physics

Noncommutativity plays an important role in modern theoretical physics. Its applications range from quantum mechanics to general relativity. Even classical mechanics has been reformulated in terms of noncommutative algebras [7].

In quantum mechanics, noncommutativity is defined by the commutator, or Lie bracket, and the commutation algebra that results from it. The commutator of two operators is defined as follows:

$$[A, B] = AB - BA, \tag{1.1}$$

and qualitatively, is a measure of the compatability of two operators. If two operators do not commute then they cannot have simultaneous eigenstates and are said to be incompatible.

The Heisenberg algebra is defined by the commutation relations

$$[x_i, p_j] = i\hbar\delta_{ij} \tag{1.2}$$

$$[x_i, x_j] = 0 \tag{1.3}$$

$$[p_i, p_j] = 0. \tag{1.4}$$

where the operators x_i and p_j are interpreted to represent the position and momentum observables respectively and δ_{ij} is the Kronecker delta, defined as $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. It is noteworthy that this simple commutation relation, along with

the assumption that x_i and p_i are Hermitian operators, describes all of the properties associated with position and momentum. In particular, it can be shown that if x_i and p_i satisfy this commutation relation then, each has a continuous spectrum, that different dimensions are independent, that x_i and p_i satisfy an indeterminacy relation given by $\Delta x \Delta p \geq \hbar/2$, and that momentum and position are generators of translations in space and of increases in momentum respectively [8].

Much work has been done recently in studying the consequences of changing or deforming the Heisenberg algebra. This area of research, known as noncommutative quantum mechanics, originated when a low-energy limit of string theory predicted Eq. (1.3) to be modified to

$$[x_i, x_j] = i\Theta_{ij} \tag{1.5}$$

where Θ_{ij} is a real, antisymmetric matrix. Noncommutativity among momentum operators has also been introduced in a similar way [9].

Another approach is to let the commutators themselves depend on operators. One such commutation algebra, known as minimal length uncertainty commutation relations [10–13], actually contains Snyder space as a special case. These commutation algebras produce more dramatic changes in the properties of position and momentum, and are more difficult to analyze. These algebras are discussed further in chapter 5.

As an example of some of the consequences of deforming the Heisenberg algebra, consider briefly Eq. (1.5). If Θ_{ij} is nonzero, as it must be for there to be any change from the standard case, then x_i and x_j cannot have all their eigenstates in common. This means that a measurement of position would vary depending upon which coordinate was measured first. Furthermore, a mathematical description of the space would not allow for x_i and x_j to be both represented by multiplicative operators since they would not satisfy the necessary commutation relations. While the changes made to the commutation algebra by Eq. (1.5) are small, the physical consequences

are profound and require drastic changes to both the mathematical description of the space as well as the intuitive understanding of it.

1.5 Snyder space as a noncommutative space

While all of these modifications to the standard commutation algebra of quantum mechanics are recent considerations, in some sense they are the natural theoretical extension of quantum mechanics. If nearly all of the properties of quantum mechanics can be deduced from the Heisenberg algebra, then in mathematical language quantum mechanics is essentially the discovery of the non-commutativity of position and momentum. One may consider what other observables do not commute. It is conceivable that there is a nontrivial commutation relation among spatial dimensions that has not been observed because it is too minute, just as quantum mechanics was not discovered until the 20th century because the effects of the factor $\hbar$ are usually relatively small. It is the task of theoretical physicists to investigate such possibilities, discriminate between promising generalizations and dead ends, and make predictions which experimentalists can put to the test.

When considering possible modifications to the Heisenberg algebra, nonzero commutators between operators acting in different dimensions is one of the simplest possibilities. The claim that quantized spacetime requires that different dimensions are no longer independent reinforces this choice. Deformed algebras of this type are the realm of noncommutative quantum mechanics.

Adding some small operator dependence to the position-momentum commutator is another way of generalizing the Heisenberg algebra. This generalization leads to minimal-length uncertainty relations and similar algebras. One can qualitatively rule out many of the possible operator dependencies in the position-momentum commuta-

tor. For example, position dependence in the commutator is unlikely, since it would imply that dynamics are location dependent. Similarly, if the dependence on momentum were linear, the commutator would depend on the direction of motion, which is also unappealing. The next simplest modification is adding a term proportional to the square of the momentum. This modification predicts modified dynamics in the high-energy regime, which is the sort of behavior we expect. This is precisely the prediction made by Snyder space.

Independently of Snyder, Kempf constructed the minimal-length uncertainty algebra, whose lowest order correction term is proportional to the square of the momentum [10]. Snyder's algebra is actually a subset of Kempf's. In addition to Kempf's work, Jaroszkiewicz derived Snyder's commutation algebra from a dynamical model that uses Dirac's constraint theory [14]. In fact, he found that by requiring that the commutator satisfy the Jacobi identity, Snyder's algebra was the unique solution to his model.

Another modified commutation algebra was proposed in the early 1980s by Saavedra, *et al.* to describe high-energy interactions, such as those between quarks [15, 16]. This modification is known as dynamical quantization, since it adds a term to the commutator of position and momentum that is proportional to the Hamiltonian. In the case of a nonrelativistic free particle this modification is also equivalent to the Snyder algebra.

Thus, Snyder space holds a special place among the theories of deformed commutation algebras. It follows the principle that spacetime must be quantized while maintaining Lorentz invariance. In addition, Snyder space holds historical significance as the first formulation of quantum mechanics on a noncommutative manifold. These reasons led me to study Snyder space as a backdrop for an exploration into generalized but reasonable models of nature.

1.6 Overview of thesis

In this work I review the fundamental properties of Snyder space and explain various mathematical techniques that I use to study it. In chapter 2, I briefly review Snyder's derivation and some of the properties of the space. In chapters 3 and 4, I consider the simple harmonic oscillator in one and two dimensions respectively, giving both exact and perturbative solutions. In chapter 5, I explain the relationship that Snyder space holds with various models of noncommutative quantum mechanics. Finally, in chapter 6, I collect my results and assess their importance and indicate possible avenues of future research. The three appendices present material whose inclusion in the main body of the text would be distracting. In appendix A, I present the scarce biographical information about Hartland Sndyer that I was able to find. In appendix B, I collect the algorithms that I developed in Maple to manipulate complex expressions and deal with noncommutativity. Appendix C summarizes results from perturbation theory used in chapter 3 and 4.

Chapter 2

Snyder Space Essentials

2.1 Review of Snyder's contribution

In this chapter, I review Snyder's derivation and outline some of the characteristics of Snyder space. As mentioned in the introduction, Snyder's motivation for developing Snyder space was far removed from the reasons for which it has been resurrected in recent years. Snyder's goal was to remove singularities from quantum field theory, but today the appeal of Snyder space is in the connection it may have with quantum gravity.

There is a certain irony in the fact that Snyder's work begins with relativistic considerations. His goal is to define position operators that were both discrete and Lorentz invariant. To accomplish this, he defines a de Sitter space embedded in a five-dimensional space,

$$-\eta^2 = \eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2, \tag{2.1}$$

where the η_i's are assumed to to range over all real numbers. This space is not the physical space in which we live, but could be interpreted as a space of projective coordinates of a real four-dimensional space. In this work, I shall refer to this space

as the *underlying space*, $\mathcal{U}$.

Snyder's next step is to define position operators, $\hat{x}_i$ and $\hat{t}$ which take the form

$$\hat{x_i} = ia\left(\eta_4\frac{\partial}{\partial\eta_i} - \eta_i\frac{\partial}{\partial\eta_4}\right), \quad i = 1, 2, 3 \tag{2.2}$$

$$\hat{t} = \frac{ia}{c}\left(\eta_4\frac{\partial}{\partial\eta_0} + \eta_0\frac{\partial}{\partial\eta_4}\right), \tag{2.3}$$

where c is the speed of light. I have denoted operators by writing a 'hat' in these expressions. Throughout this chapter I continue to denote operators in this way so as to avoid any possible confusions that may arise. In subsequent chapters, however, I dispense with this convention as it should be clear from the context if an expression is an operator or some other expression.

The operators, $\hat{x}_i$ and $\hat{t}$ defined above represent the physical observables, position and time, and are therefore assumed to be Hermitian so as to have real eigenvalues. The factor a that appears in the definition has the dimension of length and is the fundamental new ingredient of Snyder's theory. It is straightforward to show that the three spatial operators have eigenvalues ma where m is any integer, while the time operator has a continuous spectrum.

One can show that these operators are invariant under a Lorentz transformation. Consider the transformation defined by

$$\eta_0' = \gamma\left(\eta_0 + \frac{v}{c}\eta_1\right) \tag{2.4}$$

$$\eta_1' = \gamma\left(\eta_1 + \frac{v}{c}\eta_0\right), \tag{2.5}$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{1/2}. \tag{2.6}$$

Then some straightforward algebra reveals that $\eta_0'^2 - \eta_1'^2 = \eta_0^2 - \eta_1^2$, leaving quadratic form (2.1) invariant. Furthermore, the operators $\hat{x}'$ and $\hat{t}'$, which represent the operators in terms of the primed coordinates can be related to $\hat{x}$ and $\hat{t}$, in terms of the

unprimed coordinates:

$$\hat{x}' = \gamma\left(\hat{x} - v\hat{t}\right) \tag{2.7}$$

$$\hat{t}' = \gamma\left(\hat{t} - \frac{v}{c^2}\hat{x}\right). \tag{2.8}$$

Therefore, the operators defined in Eqs. (2.2) and (2.3) are invariant under Lorentz transformations.

In addition, the four momentum operators are defined as

$$\hat{p_i} = (\hbar/a)(\eta_i/\eta_4), \quad i = 0, 1, 2, 3 \tag{2.9}$$

and can be shown to have continuous spectra. This definition generates the modified commutation relations:

$$[\hat{x_i}, \hat{p_j}] = i\hbar\left(\delta_{ij} + \alpha\hat{p_i}\hat{p_j}\right), \quad i, j = 1, 2, 3 \tag{2.10}$$

$$[\ \hat{t},\ \hat{p_t}] = i\hbar\left(1 - \alpha\hat{p_t}^2\right) \tag{2.11}$$

$$[\hat{x_i}, \hat{p_t}] = c^2[\hat{p_i},\ \hat{t}] = i\hbar\alpha\hat{p_i}\hat{p_t}, \qquad i = 1, 2, 3, \tag{2.12}$$

where we have introduced the constant $\alpha = (a/\hbar)^2$, which has dimensions of inverse momentum squared. It is clear that in the limit that a approaches zero we recover the canonical commutation relations from standard quantum mechanics.

In addition to the position and momentum operators, Snyder also defines angular momentum operators in the usual way:

$$\hat{L}_k = \epsilon_{ijk}\hat{x}_i\hat{p}_j, \tag{2.13}$$

where ϵ_{ijk} is the completely antisymmetric unit tensor. In addition, Snyder also defines the generators of boosts,

$$\hat{M}_i = \frac{1}{c}\hat{x}_i\hat{p}_t + c\hat{t}\hat{p}_i. \tag{2.14}$$

Algebraic manipulations reveal that these operators have a representation in the underlying space given by

$$\hat{L}_x = i\hbar\left(\eta_3\frac{\partial}{\partial\eta_2} - \eta_2\frac{\partial}{\partial\eta_3}\right) \tag{2.15}$$

$$\hat{M}_x = i\hbar\left(\eta_0\frac{\partial}{\partial\eta_1} + \eta_1\frac{\partial}{\partial\eta_0}\right), \tag{2.16}$$

with similar expressions for $\hat{L}_y, \hat{L}_z, \hat{M}_y$, and $\hat{M}_z$.

The commutator between coordinates for different dimensions can be written in terms of the angular momentum operators and the generators of boosts,

$$[\hat{x}_i, \hat{x}_j] = i\hbar\alpha\epsilon_{ijk}\hat{L}_k, \tag{2.17}$$

$$[\hat{x}_i, \hat{t}] = \frac{i\hbar}{c}\alpha\hat{M}_i \tag{2.18}$$

One of the differences between Snyder space and standard quantum mechanics is the way in which time enters the formulation. Snyder space incorporates time as an operator. This is a natural consequence of incorporating Lorentz invariance in the formulation since Lorentz transformations require space and time to take equivalent roles. In standard quantum mechanics time is a parameter as in classical mechanics, and complications arise when one tries to introduce a time operator to standard quantum mechanics. I will not consider any time dependence in this work, but it should be noted that Snyder space allows for the natural introduction of a time operator.

In spite of the mentioned differences between Snyder space and standard quantum mechanics, they share some important properties. For example, the commutation relations among the angular momentum operators remain unchanged, as does their definition in terms of the position and momentum operators. Thus, the orbital angular momentum spectrum is unchanged. Snyder does not attempt to incorporate spin into his model, and it is uncertain whether spin effects would be the same as in standard

quantum mechanics. The fact that Snyder formulates the space in such a way to maintain as many of these characteristics as possible is one of the most appealing aspects of Snyder space. In particular, since there is no apparent discrepancy between observation and the current theory, a model for quantum gravity should be as closely related to standard quantum mechanics as possible.

2.2 The underlying space

A striking characteristic of Snyder space is the existence of the underlying space, $\mathcal{U}$. Later in this chapter I look at the momentum space representation of Snyder space. In principle, all calculations done in Snyder space can be done in the underlying space or the momentum space representations, thus Snyder space has the advantage of two levels for corroboration. In this section I look more closely at the underlying space and its properties.

Since throughout this thesis I will consider systems with fewer than four dimensions, I begin by writing down the one-dimensional analog of Eq. (2.1), along with the position and momentum operators,

$$-\eta^2 = -\eta_1^2 - \eta_4^2 \tag{2.19}$$

$$\hat{x} = ia\left(\eta_4 \frac{\partial}{\partial \eta_1} - \eta_1 \frac{\partial}{\partial \eta_4}\right) \tag{2.20}$$

$$\hat{p} = \frac{\hbar}{a}\left(\frac{\eta_1}{\eta_4}\right). \tag{2.21}$$

By making a simple change of variables we can reduce the two-variable dependence on η_1 and η_4 to a single, periodic variable, ϕ. By defining polar-like coordinates, still in the underlying space,

$$\eta_1 = \eta\ \sin(\phi) \tag{2.22}$$

$$\eta_4 = \eta\ \cos(\phi), \tag{2.23}$$

position and momentum take the simple form

$$\hat{x} = ia\,\frac{\partial}{\partial\phi} \tag{2.24}$$

$$\hat{p} = \frac{\hbar}{a}\,\tan(\phi). \tag{2.25}$$

Note that the η of Eqs. (2.22) and (2.23) is the same as the η of Eq. (2.19), but it has disappeared in Eqs. (2.24) and (2.25). The η of Eq. (2.19) is a constant that defines the curvature of the underlying de Sitter space. Thus the results of Snyder space are independent of η.

The discrete spectrum of the position operator follows from requiring that its eigenfunctions be single-valued. In this case, the normalized eigenstates $|m\rangle$ correspond to eigenfunctions with eigenvalues ma in the ϕ representation which take the form

$$\langle\phi|m\rangle = \frac{1}{\sqrt{2\pi}}e^{-im\phi}, \tag{2.26}$$

where m ranges over all integers. Thus, a Fourier decomposition of a wave function results in the representation in the position eigenfunction basis.

In two dimensions, the underlying space is defined as follows:

$$-\eta^2 = -\eta_1^2 - \eta_2^2 - \eta_4^2 \tag{2.27}$$

$$\hat{x} = ia\left(\eta_4\frac{\partial}{\partial\eta_1} - \eta_1\frac{\partial}{\partial\eta_4}\right) \tag{2.28}$$

$$\hat{y} = ia\left(\eta_4\frac{\partial}{\partial\eta_2} - \eta_2\frac{\partial}{\partial\eta_4}\right) \tag{2.29}$$

$$\hat{L}_z = i\hbar\left(\eta_2\frac{\partial}{\partial\eta_1} - \eta_1\frac{\partial}{\partial\eta_2}\right) \tag{2.30}$$

$$\hat{p}_x = \frac{\hbar}{a}\left(\frac{\eta_1}{\eta_4}\right) \tag{2.31}$$

$$\hat{p}_y = \frac{\hbar}{a}\left(\frac{\eta_2}{\eta_4}\right). \tag{2.32}$$

As before, we introduce angular coordinates, this time spherical polar coordiantes. The three operators $\hat{x}$, $\hat{y}$, and $\hat{L_z}$ represent the generators of rotations about the three

axes η_2, η_1, and η_4 respectively. In choosing how to define the spherical polar coordinates, one must choose a "preferred axis." In standard quantum mechanics the three rotations are generated by the three components of angular momentum and the z-component is arbitrarily chosen as the preferred component. In our case we must choose one of the three operators $\hat{x}$, $\hat{y}$, or $\hat{L}_z$ as the preferred component. The actual choice of coordinates will depend on the symmetry of the system under consideration. If the system happens to possess radial symmetry then $\hat{x}$ and $\hat{y}$ appear symmetrically in the system and the natural choice is $\hat{L}_z$. Furthermore, as we will see, $\hat{L}_z$ commutes with any radially symmetric Hamiltonian, and, therefore, has simultaneous eigenstates. If the system does not possess radial symmetry, then there may be another natural choice determined by the symmetry of the system. With this symmetry in mind, we make the change of variables given by

$$\eta_1 = \eta \ \sin(\theta) \ \cos(\phi) \tag{2.33}$$

$$\eta_2 = \eta \ \sin(\theta) \ \sin(\phi) \tag{2.34}$$

$$\eta_4 = \eta \ \cos(\theta), \tag{2.35}$$

from which it follows that

$$\hat{x} = ia\left(-\sin(\phi)\frac{\partial}{\partial\theta} - \cot(\theta)\cos(\phi)\frac{\partial}{\partial\phi}\right) \tag{2.36}$$

$$\hat{y} = ia\left(\cos(\phi)\frac{\partial}{\partial\theta} - \cot(\theta)\sin(\phi)\frac{\partial}{\partial\phi}\right) \tag{2.37}$$

$$\hat{L}_z = i\hbar\frac{\partial}{\partial\phi} \tag{2.38}$$

$$\hat{p}_x = \frac{\hbar}{a}\tan(\theta)\cos(\phi) \tag{2.39}$$

$$\hat{p}_y = \frac{\hbar}{a}\tan(\theta)\sin(\phi). \tag{2.40}$$

Using these definitions, the non-relativistic kinetic energy, $\hat{T}$, is given by

$$\hat{T} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} = \frac{\hbar^2\tan^2(\theta)}{2ma^2}. \tag{2.41}$$

Thus, the mathematical form of the kinetic energy in two dimensions is nearly identical to that in one dimension. Furthermore, it immediately follows that $\hat{L}_z$ commutes with total kinetic energy, $\hat{T}$, since the total kinetic energy does not depend on θ.

The other operators defined in Eqs. (2.36) through (2.38) are a bit more complicated. Since we made the definition in Eqs. (2.33) - (2.35) assuming radial symmetry, we define the square of the radial position operator by

$$\hat{r}^2 = \hat{x}^2 + \hat{y}^2. \tag{2.42}$$

It is straightforward to show that the commutator of the position operators and the angular momentum operators are the same as in standard quantum mechanics. Thus, using

$$[\hat{x}, \hat{L}_z] = -i\hbar\hat{y} \tag{2.43}$$

$$[\hat{y}, \hat{L}_z] = \ i\hbar\hat{x}, \tag{2.44}$$

we have that the radial position operator commutes with the angular momentum:

$$[\hat{r}^2, \hat{L}_z] = \hat{x}[\hat{x}, \hat{L}_z] + [\hat{x}, \hat{L}_z]\hat{x} + \hat{y}[\hat{y}, \hat{L}_z] + [\hat{y}, \hat{L}_z]\hat{y} \tag{2.45}$$

$$= -i\hbar\left(\hat{x}\hat{y} + \hat{y}\hat{x} - \hat{y}\hat{x} - \hat{x}\hat{y}\right) \tag{2.46}$$

$$= 0. \tag{2.47}$$

Combining this result with the fact that $\hat{L}_z$ commutes with the kinetic energy, $\hat{T}$, it follows that the angular momentum commutes with any radially symmetric Hamiltonian. Thus, in two dimension, the operators $\hat{H}$ and $\hat{L}_z$ represent a complete set of commuting observables.

In addition, we can consider the following combination:

$$\frac{\hat{r}^2}{a^2} + \frac{\hat{L}_z^2}{\hbar^2} = -\left(\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right). \tag{2.48}$$

The eigenfunctions of this operator are the well-known spherical harmonics. The eigenvalues are given by $l(l+1)$, where l is a non-negative integer. Since the eigenvalues of angular momentum are $\hbar m$, where $-l < m < l$, it follows that the eigenvalues of the square of the radial position operator are $a^2\left(l(l+1) - m^2\right) = na^2$, where n is any non-negative integer.

In a single dimension, we can expand any wave function in the position eigenfunction basis. In two dimensions, this is not possible. Since $\hat{x}$ and $\hat{y}$ do not commute, one cannot assign simultaneous eigenvalues to both observables and, consequently, a basis cannot be formed from the tensor product of the basis of $\hat{x}$ and the basis of $\hat{y}$.

While the underlying space $\mathcal{U}$ is the fundamental element of Snyder space, many consider the existence of the underlying space, and in particular Snyder's ad hoc introduction of the underlying space, to be the least appealing aspect of the formulation [14]. Furthermore, it is difficult to glean any immediate physical interpretation from wave functions in the underlying space. Thus, the underlying space is usually not the ideal framework in which to perform calculations.

Since momentum is a multiplicative operator in the underlying space, it is easy to formulate a momentum space solution to the commutation relations. A momentum space representation would have the advantage of being more closely related to standard quantum mechanics. Hopefully many of the mathematical techniques of standard quantum mechanics will also be applicable to Snyder space. In addition, momentum is a physically relevant quantity, especially in high-energy systems, so a momentum space representation is more indicative of the physical situation than the abstract underlying space. We will discuss the momentum space representation in the next section.

In contrast, we do not expect to find a position space representation in which momentum is represented in terms of derivatives, since position has a discrete spectrum.

Furthermore, since the position operators do not commute, they cannot simultaneously be represented by multiplicative operators. The most common basis representation used in quantum mechanics is position space. Since many of the concepts studied in standard quantum mechanics are interpreted through the position space representation, these concepts become much more difficult to understand in Snyder space. Some examples of these complications are discussed below in section 2.6.

2.3 Momentum space

In one dimension, the transition from the underlying space to momentum space is straightforward. In both spaces, the momentum operator is multiplicative so the transition is accomplished through a simple change of variables that transforms the momentum operator into multiplication by p

$$\hat{p} = \frac{\hbar}{a}\tan(\phi) = p\mathbf{1} = p \ \cdot . \tag{2.49}$$

Then solving the transformation for ϕ and substituting in Eq. (2.24), the position operator becomes

$$\hat{x} = i\hbar\left(1 + \alpha p^2\right)\frac{\partial}{\partial p}. \tag{2.50}$$

In the limit that a, and therefore α, approaches zero while holding $\hbar$ constant, these expressions become the momentum space representation of standard quantum mechanics.

As Snyder discusses in his original work, it appears that position operator as defined in Eq. (2.50) is not Hermitian. However, it should be remembered that these operators are acting on the underlying space which is related to the momentum space through a change of variables. The inner product between two wave functions requires a factor of the Jacobian of that transformation in the volume element. In

one dimension, the correct volume element is

$$d\tau = \frac{a\ dp}{\hbar\left(1+\alpha p^2\right)}. \tag{2.51}$$

This modified volume element causes the operator $\hat{x}$ to be Hermitian. Hermiticity is an important subject in choosing the correct representation. It will be discussed further in chapter 3.

To generalize momentum space to two dimensions, we proceed as before by making the change of variables

$$p_x = \frac{\hbar}{a}\ \tan(\theta)\ \cos(\phi) \tag{2.52}$$

$$p_y = \frac{\hbar}{a}\ \tan(\theta)\ \sin(\phi). \tag{2.53}$$

The new position operators are

$$\hat{x} = i\hbar\left(\frac{\partial}{\partial p_x} + \alpha p_x\left(p_x\frac{\partial}{\partial p_x} + p_y\frac{\partial}{\partial p_y}\right)\right) \tag{2.54}$$

$$\hat{y} = i\hbar\left(\frac{\partial}{\partial p_y} + \alpha p_y\left(p_y\frac{\partial}{\partial p_y} + p_x\frac{\partial}{\partial p_x}\right)\right). \tag{2.55}$$

The z-component of angular momentum is then

$$\hat{L}_z = i\hbar\left(p_y\frac{\partial}{\partial p_x} - p_x\frac{\partial}{\partial p_y}\right). \tag{2.56}$$

The two-dimensional volume element that makes these operators Hermitian is

$$d\tau = \frac{a\ dp_x\ dp_y}{\hbar\left(1+\alpha p_x^2+\alpha p_y^2\right)^{3/2}}. \tag{2.57}$$

2.4 Quantization in the infinite square well analogy

A good way to qualitatively understand the nature of the quantization of space in Snyder space is to consider an analogy with a well-known result from standard quantum mechanics. The infinite square well solution is essentially a free particle with

Figure 2.1 Restricting the range of available position values results in a discrete momentum spectrum.

boundary conditions that lead to a discrete spectrum. The energy levels go as the square of the quantum number. The quantum number ranges over positive integers. Since the energy is just the kinetic energy, the magnitude of the momentum is also discrete and is directly proportional to the quantum number. Thus, the spacing between accessible momentum values is constant. In particular, if the one-dimensional square well has a length of L, then the energy spectrum is given by

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2mL}, \quad n = 1, 2, 3, ... \tag{2.58}$$

and the allowed momenta are

$$p = \pm \frac{\hbar \pi n}{\sqrt{L}} = \pm p_0 n \tag{2.59}$$

where p_0 is the spacing between momenta. The eigenvalue for zero momentum is excluded because there is no zero-energy state in the infinite square well. These properties are summarized in Fig. 2.1.

In standard quantum mechanics, position and momentum have complementary

roles. Thus, restricting the possible values of momentum and choosing the correct linear combination of position eigenfunctions will quantize the possible values of position. In particular, if the momentum coordinate is restricted such that $-(1/2)\ p_{max} \leq p \leq (1/2)\ p_{max}$, then solving the position eigenvalue problem with the new boundary conditions results in the discrete spatial spectrum

$$x = \pm \frac{\hbar \pi n}{p_{max}} = \pm x_0 n \tag{2.60}$$

where $x_0 = \hbar\pi / p_{max}$.

It is possible to construct a momentum operator $\hat{p_S}$ that has a continuous spectrum that ranges over all real numbers:

$$\hat{p}_S = \frac{p_{max}}{\pi} \tan\left(\frac{\pi}{p_{max}} p\right) = \frac{\hbar}{x_0} \tan\left(\frac{x_0}{\hbar} p\right). \tag{2.61}$$

For small values of momentum (or equivalently, large values of p_{max} and small values of x_0), $\hat{p_S}$ and $\hat{p}$ have essentially the same behavior. Now if we associate the physical momentum with $\hat{p_S}$ and let $x_0 = a$, we will have reproduced one-dimensional Snyder space. In particular, since $\hat{x}$ and $\hat{p}$ obey the canonical commutation relations, it follows that

$$[\hat{x}, \hat{p}_S] = i\hbar \left(1 + \alpha \hat{p}_S^2\right). \tag{2.62}$$

This trick only works in one dimension. If the same procedure were applied in two dimensions, both position operators become quantized, but the correct commutation relations do not result. In particular, the relation $[\hat{x}, \hat{y}] = 0$ holds as in standard quantum mechanics. Although the analogy presented in this section falls short in higher dimensions, it is helpful for understanding the relationship between Snyder space and standard quantum mechanics.

2.5 Free particle

The simplest system that I will consider in this work is that of the nonrelativistic free particle in one dimension. A solution exists in both the underlying space and the momentum space. The solution can also be expanded in terms of the position eigenfunctions.

In the underlying space using the notation of Eqs. (2.24) and (2.25), the equation for the free particle is

$$\langle\phi|\hat{H}|\psi\rangle \;=\; \langle\phi|\frac{\hat{p}^2}{2\mu}|\psi\rangle \;=\; \frac{\hbar^2}{2a^2\mu}\tan^2(\phi)\langle\phi|\psi\rangle \;=\; E\langle\phi|\psi\rangle. \tag{2.63}$$

This equation becomes

$$\left(\frac{\hbar^2}{2a^2\mu}\tan^2(\phi) - E\right)\langle\phi|\psi\rangle = 0, \tag{2.64}$$

from which it immediately follows that

$$\langle\phi|\psi\rangle \;=\; \delta\left(\phi - \tan^{-1}\frac{a}{\hbar}\sqrt{2\mu E}\right). \tag{2.65}$$

The result is very similar in momentum space

$$\langle p|\psi\rangle \;=\; \delta\left(p - \sqrt{2\mu E}\right). \tag{2.66}$$

Thus the momentum representation of the wave function of the free particle in Snyder space is the same as that in standard quantum mechanics.

We can express this result in terms of the position eigenfunctions using the relation

$$\delta(\phi) = \sum_{m=-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{im\phi} = \sum_{m=-\infty}^{\infty}\langle\phi|m\rangle \tag{2.67}$$

where $|m\rangle$ is the eigenstate of the position operator with eigenvalue ma. Thus, we have,

$$|\psi\rangle = \sum_{m=-\infty}^{\infty} e^{-i\,\tan^{-1}\frac{a}{\hbar}\sqrt{2\mu E}}|m\rangle \tag{2.68}$$

In two dimensions, the problem is similar. Thus, we have

$$\langle\theta,\phi|\hat{H}|\psi\rangle \;=\; \langle\theta,\phi|\frac{\hat{p}^2+\hat{q}^2}{2\mu}|\psi\rangle \;=\; \frac{\hbar^2}{2a^2\mu}\tan^2(\theta)\langle\theta,\phi|\psi\rangle \;=\; E\langle\theta,\phi|\psi\rangle. \tag{2.69}$$

From which it follows that

$$\langle\theta,\phi|\psi\rangle \;=\; \delta\left(\theta-\tan^{-1}\left(\frac{a}{\hbar}\sqrt{2\mu E}\right)\right). \tag{2.70}$$

2.6 Particles in potentials

Generalizing the results of the previous section to include particles in various potentials presents a number of complications. In most introductory texts on quantum mechanics, a study of the free particle is followed by one on particles in piecewise constant potentials, such as the square well. The problem with these potentials in Snyder space is that we are restricted to working with representations in the underlying space or the momentum space. In both cases, the potential is defined as a piecewise function of a differential operator, resulting in an untractable differential equation. For this reason, I will restrict my attention to potentials that are analytic functions of the position. However, methods involving operator transformations discussed in chapters 3 and 4 may be applied to solve such systems. For example, using these methods I find the energy spectrum and wave functions of the one-dimensional infinite square well in section 5.5.3.

The simplest analytic function that one might consider is the linear potential. The monotonic linear potential does not have bound states, however, and in this thesis I will be studying how the energy levels of bound states are altered in Snyder space. The simplest system with bound states is the quadratic potential or Simple Harmonic Oscillator (SHO), which has a variety of applications in quantum mechanics. The next two chapters will focus on the SHO in one and two dimensions. Other

potentials which could be the subject of future research, include the higher polynomial potentials, such as cubic or quartic potentials, or inverse powers, such as the Coulomb potential. Analytic solutions could exist for these potentials, but we expect that they are more difficult to find than those of the corresponding potential in standard quantum mechanics. The potential given by the absolute value of the position, i.e. $V(x) = |x|$, falls in the class of potentials defined as piecewise functions and exhibits the problems discussed in the previous paragraph.

Since Snyder space is an alternate formulation of quantum mechanics, all of the usual applications considered in standard quantum mechanics are potential subjects of study in Snyder space. In particular, research questions include such things as "What are the modified energy spectrum and wavefunctions in Snyder space?", "How do concepts such as spin, position, momentum, and angular momentum change in Snyder space?", "What concepts remain the same as in standard quantum mechanics?", and "Are there any elements of Snyder space that are drastically different from standard quantum mechanics?" I will address these questions briefly in this thesis; however, a thorough analysis is beyond the scope of this work.

Chapter 3

One-Dimensional Simple Harmonic Oscillator

3.1 Operator transformations

In a single dimension the Snyder space commutation relations reduce to the single commutation relation

$$[x, p] = i\hbar \left(1 + \alpha p^2\right), \tag{3.1}$$

and the Hamiltonian for the Simple Harmonic Oscillator (SHO) is

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2. \tag{3.2}$$

Note that I have dispensed with the convention of chapter 2 in which I denote operators by a 'hat' over the symbol of the operator. It should be clear from the context which expressions are operators.

There are several ways to analyze this system. One possible method is to apply the momentum-space representation of the position and momentum operators on a wave function and solve the eigenvavlue problem in the form of a differential equation. This

method is very general and, as we have seen, can always be applied to any system, as long as the potential energy is an analytic function of the position.

For the SHO, however, there are other methods. In standard quantum mechanics, the SHO eigenvalue problem can be solved algebraically using operator methods. While it is possible to do the same thing in Snyder space [13], it is a bit more complicated.

Approaching the problem qualitatively, since the effects of Snyder space are determined by the fundamental length a which is assumed to be small, we should expect the deviation from the standard result to be likewise small. We can therefore consider Snyder space to be a perturbation of standard quantum mechanics and the resulting spectrum to be obtained perturbatively from the standard spectrum.

To formulate the Snyder space simple harmonic oscillator as a perturbed SHO in standard quantum mechanics, we borrow a technique from noncommutative quantum mechanics. We will make an operator transformation relating the Snyder space position and momentum operators to the canonical operators from quantum mechanics obeying the Heisenberg algebra [17]. In particular, denoting the Snyder space operators with a subscript S, we have

$$x_S = x_S(x, p) \tag{3.3}$$

$$p_S = p_S(x, p), \tag{3.4}$$

where x and p obey the canonical commutation relations whereas x_S and p_S obey the commutation relation given by Eq. (3.1).

An important point to observe about Eqs. (3.3) - (3.4) is that x and p are not the standard operators of quantum mechanics, although they may appear to be very similar. One reason for making this distinction, is that x and p are no longer required to be Hermitian, and as such they do not correspond to physical observables. On the

other hand, x_S and p_S are Hermitian and correspond to the observable position and momentum. In fact, the question of hermiticity is crucial to choosing an operator transformation and will be discussed in the next section.

The proposed method will fail unless we can find solutions to Eqs. (3.4) - (3.3). It is easy to verify that

$$\begin{aligned} x_S &= x + \alpha p\, x\, p \\ p_S &= p \end{aligned} \tag{3.5}$$

satisfy Eqs. (3.3) - (3.4). However, this solution is not unique, since, for example, it can also be shown that

$$\begin{aligned} x_S &= x + \alpha \left(t\, p^2\, x + (1-t) x\, p^2 \right), \quad 0 \leq t \leq 1 \\ p_S &= p, \end{aligned} \tag{3.6}$$

is a solution, where t ranges continuously on the given range. Yet another is given by

$$\begin{aligned} x_S &= x \\ p_S &= \frac{1}{\sqrt{\alpha}} \tan\left(\sqrt{\alpha} p\right). \end{aligned} \tag{3.7}$$

There may be other transformations, but they are likely to be more complicated. Since there is not a unique transformation, it is not immediately clear which transformation should be used. This question is addressed is in section 3.2.

Since our goal is to apply perturbation theory to the standard SHO spectrum, our next step is to relate x and p to a and $a^\dagger$, the raising and lowering operators, given in standard quantum mechanics by

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{i}{m\omega} p \right) \tag{3.8}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{i}{m\omega} p \right). \tag{3.9}$$

In order for these two operators to have the same functionality as they do in standard quantum mechanics, it is necessary for them to be Hermitian conjugates of one another. Therefore, the question regarding the hermiticity of x and p becomes very important. While the hermiticity condition on x and p is not automatically satisfied by the definition in Eqs. (3.3) - (3.4), the commutation relation $[a, a^\dagger] = 1$, which is also critical for obtaining the spectrum, is automatically satisfied due to the fact that x and p satisfy the canonical commutation relations.

This method of introducing the operators x and p has both advantages and disadvantages. In standard quantum mechanics, the simple harmonic oscillator can be solved exactly using algebraic operator methods; introducing similar operators in Snyder space results in a system to which perturbation theory can be easily applied. Furthermore, the final solution will be independent of any particular representation. However, by not working in any particular basis, there is some degree of ambiguity in the definition of x and p. Simply postulating the commutation relation between x and p does not uniquely define either operator, as can be seen from the many solutions to Eqs. (3.3) - (3.4). It is unclear at this stage whether the added condition that $a^\dagger$ is the Hermitian conjugate of a resolves the ambiguity; however, the issue is discussed in the next section.

3.2 Hermiticity

In order to address the issue of hermiticity, we must return to the underlying space. The inner product between any two states, $|\chi\rangle$ and $|\chi'\rangle$, is given by Snyder in [2] to be

$$\langle\chi'|\chi\rangle = \int \frac{d\mathbf{p}\ dp_t}{D(p_t, \mathbf{p})} \chi'^*(\mathbf{p}, p_t)\chi(\mathbf{p}, p_t), \tag{3.10}$$

where $D(\mathbf{p}, p_t)$ is a factor that makes the position operators Hermitian. It is essentially the volume element of the hyper-surface $\eta^2 = constant$. In a single dimension, $D(\mathbf{p}, p_t) = \frac{(1+\alpha p^2)}{\sqrt{\alpha}}$, which is just the Jacobian of the transformation encountered in Eq. (2.49).

Given the definition in Eq. (3.10), the Hermitian conjugate of an operator $\mathcal{L}$, denoted by $\mathcal{L}^\dagger$, is given in [18] by the relation

$$\langle \chi' | \mathcal{L} | \chi \rangle = \langle \chi | \mathcal{L}^\dagger | \chi' \rangle^*, \tag{3.11}$$

where the star denotes the complex conjugate.

In order to apply the definition of hermiticity to x and p, we must decide on some representation of x and p in Snyder's momentum space. Since our goal is to construct operators closely related to the position and momentum operators of standard quantum mechanics, we will let x denote the differential operator and let p represent multiplication by the momentum. Using this definition, it seems that the correct transformation to choose is

$$x_S = (1 + \alpha\, p^2)\, x \tag{3.12}$$

$$p_S = p, \tag{3.13}$$

since this representation corresponds exactly to the momentum space definition of x_S and p_S given in Eqs. (2.49) - (2.50). While this produces the expected result, we will shortly see that it still does not uniquely define the correct transformation. Furthermore, there may be other transformations that are more convenient.

A second candidate for a viable transformation is given by

$$x_S = x + \alpha p\, x\, p \tag{3.14}$$

$$p_S = p. \tag{3.15}$$

At first it may seem that this transformation does not meet the hermiticity condition, since the physical position operator would then have the momentum space representation

$$x_S = i\hbar\left(\frac{\partial}{\partial p} + \alpha p\frac{\partial}{\partial p}p\right), \tag{3.16}$$

which appears not to be Hermitian with respect to the correct volume element. However, this operator is certainly Hermitian with respect to the volume element of standard quantum mechanics, dp. Thus, if we consider an inner product between two vectors χ and χ' in one dimension, we have

$$\langle\chi'|\chi\rangle = \int \frac{dp\sqrt{\alpha}}{(1+\alpha p^2)}\chi'^*(p)\chi(p) = \int dp\ \tilde{\chi}'^*(p)\ \tilde{\chi}(p), \tag{3.17}$$

where

$$\tilde{\chi}'(p) = \sqrt{\frac{\sqrt{\alpha}}{(1+\alpha p^2)}}\chi'(p) \tag{3.18}$$

$$\tilde{\chi}(p) = \sqrt{\frac{\sqrt{\alpha}}{(1+\alpha p^2)}}\chi(p). \tag{3.19}$$

Using this definition, $\tilde{\chi}$ has the advantage that its square modulus is the probability amplitude, which is the physically relevant quantity. Furthermore, by absorbing the factor of $D(p)$ into the wave function, the relevant volume element is simply dp and the operator given by Eq. (3.16) is Hermitian. In addition, one can show that

$$\int \frac{dp\sqrt{\alpha}}{(1+\alpha p^2)}\chi'^*(p)(i\hbar)\left(1+\alpha p^2\right)\frac{\partial}{\partial p}\chi(p) = \int dp\tilde{\chi}'^*(p)(i\hbar)\left(\frac{\partial}{\partial p} + \alpha p\frac{\partial}{\partial p}p\right)\tilde{\chi}(p). \tag{3.20}$$

This suggests that if we require that a and $a^\dagger$ be Hermitian conjugates of one another and use the transformation $p_S = p$ and $x_S = x + \alpha pxp$, after applying perturbation theory to the system, the resulting wave functions will be related to the wave functions obtained through the transformation $p_S = p$ and $x_S = (1+\alpha p^2)x$ by a factor of $\sqrt{\sqrt{\alpha}/(1+\alpha p^2)}$. Furthermore, the resulting energy spectrum will be

identical under both transformations. Whereas the reasoning above does not amount to a rigorous proof, it corresponds precisely to what we find in subsequent sections.

In this section we have only considered two possible solutions to Eqs. (3.3) - (3.4). There are many other possible transformations given in Eqs. (3.6) - (3.7). The two solutions considered in this section were chosen because of their close relation to the momentum representation of the position operators both in Snyder space and in standard quantum mechanics. The arguments of this section suggest that any transformation will result in the correct energy spectrum and in a wave function related to the actual wave function by some other transformation. Again, we will see that this qualitative argument is corroborated for at least one more transformation: $p_S = p$ and $x_S = x(1 + \alpha p^2)$.

In summary, in order to apply perturbation theory to the SHO by means of the operators a and $a^\dagger$, we require that they be Hermitian conjugates of one another, which requires that x and p each be Hermitian operators as well. Finally, x_S and p_S must be Hermitian since they represent physical observables. All three of these conditions are met by only two of the transformations discussed above, the transformation given in Eq. (3.5) and in Eq. (3.7). These are the transformations that we will use in our analysis of the SHO.

3.3 Perturbation treatment

Having addressed the problem of the uniqueness of the operator transformations, we can now easily apply perturbation theory. I briefly review the principles of perturbation theory in quantum mechanics in appendix C, where I also give notation conventions. For a more detailed treatment, the reader is referred to any text on quantum mechanics, see for example [19].

In the case of the SHO of standard quantum mechanics, the Hamiltonian takes the form

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right), \tag{3.21}$$

with eigenvalues given by

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \tag{3.22}$$

where n ranges over the natural numbers. The corresponding eigenstates are orthogonal and can be normalized:

$$\langle m|n\rangle = \delta_{mn}. \tag{3.23}$$

The action of the raising and lowering operators

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{3.24}$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle, \tag{3.25}$$

is sufficient to calculate the corrections to the simple harmonic oscillator energy spectrum in Snyder space.

If we take the transformation $p_S = p$ and $x_S = x + \alpha pxp$ and apply the definition of the raising and lowering operators, then the Hamiltonian takes the form

$$\begin{aligned} H &= \frac{p_s^2}{2m} + \frac{1}{2}m\omega^2 x_s^2 \\ &= \hbar\omega(a^\dagger a + \frac{1}{2}) \\ &\quad + \frac{1}{4}\hbar^2\omega^2\alpha m\left(2\,a^{\dagger 2}a^2 + 4\,a^\dagger a + 1 - a^{\dagger 4} - a^4\right) \\ &\quad + \frac{1}{16}\hbar^3\omega^3\alpha^2 m^2\left(7 + 26\,a^\dagger a + 18\,a^{\dagger 2}a^2 + 4\,a^{\dagger 3}a^3\right) \\ &\quad + \frac{1}{16}\hbar^3\omega^3\alpha^2 m^2\left(-7\,a^2 - 7\,a^{\dagger 2} - 5\,a^{\dagger 4} - 5\,a^4 - 4\,a^\dagger a^3 - 4\,a^{\dagger 3}a\right) \\ &\quad + \frac{1}{16}\hbar^3\omega^3\alpha^2 m^2\left(a^6 + a^{\dagger 6} - 2\,a^{\dagger 5}a - 2\,a^\dagger a^5 - a^{\dagger 2}a^4 - a^{\dagger 4}a^2\right). \end{aligned} \tag{3.26}$$

While the method to arrive at Eq. (3.26) is straightforward, the actual calculations are tedious. I used algorithms that I had previously developed precisely to manipulate this type of expression. The actual algorithms are given in appendix B.1.3.

It should be noted that the Hamiltonian given in Eq. (3.26) is exact. No approximations have been made to this point. In terms of perturbation theory, we consider the perturbation parameter to be α, or rather, the dimensionless quantity $\alpha m\omega\hbar$. As such, the resulting perturbation is unusual in that the perturbation term contains both first and second order terms of the perturbation parameter. In practice, we will write Eq. (3.26) as

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) + \lambda V, \tag{3.27}$$

where V includes all the perturbation terms. After we apply the formulas from perturbation theory, we set $\lambda = 1$ and collect terms in powers of α. For consistency, we discard contributions that are of higher order in α than the order of the perturbation calculation. For example, when we calculate the second-order perturbation correction, we obtain terms proportional to α^3 and α^4. These terms are meaningless, however, since other contributions in the same powers of α that would appear when calculating third-order perturbation corrections have not been accounted for. In other words, if we had calculated the third or fourth-order perturbation theory, there would have been additional third and fourth-order contributions in α; therefore, we discard all terms involving powers of α higher than two in a second-order calculation.

In principle, perturbation calculations can be done to any order; however, the increasing complexity of the formulas make them computationally expensive. Furthermore, as we will see in section 3.6, the size of the perturbation is very small, so higher order corrections are not likely to be significant. In addition to the algorithms we developed to manipulate noncommutative operators, we also developed algorithms to evaluate successive orders of perturbation theory. These are included and discussed in appendix B.2. We implemented the algorithms to evaluate fourth

order perturbation corrections. The resulting energy spectrum is given by

$$E_n = \hbar\omega\left(\left(n+\frac{1}{2}\right)+\frac{1}{4}\beta\left(2n^2+2n+1\right)+\left(n+\frac{1}{2}\right)\left(\frac{1}{8}\beta^2-\frac{1}{128}\beta^4\right)\right), \qquad (3.28)$$

where we have introduced the dimensionless parameter $\beta = \alpha m\omega\hbar$.

The perturbed energy spectrum is identical for each transformation we consider. In fact, our original motivation for calculating the energies to fourth order to uncover discrepancies in the energy spectra proved to be misguided and the hermiticity argument vindicated.

There are several noteworthy characteristics about the energy spectrum given above. First, there is the peculiar recurrence of terms proportional to $(n + 1/2)$. Also, the third order term vanishes. These characteristics are striking and suggest the possible existence of a closed form expression for the exact energy. One is tempted to conjecture that the fifth-order perturbation will vanish along with all other successive odd-orders and that the sixth-order and term and all successive even-order contributions will be proportional to $(n + 1/2)$. In the next few sections we will see that this is the case.

3.4 Exact treatment

The results in the previous section demonstrate that, although the process is laborious, the spectrum of the Snyder SHO can be found perturbatively. In the current section we establish an exact solution and show the consistency between exact and perturbative solutions.

Applying the definition of the position and momentum operators in both the underlying space and in momentum space to the Hamiltonian eigenvalue problem results in a second-order differential equation. It turns out that in both cases the differential equation has solutions in terms of the associated Legendre functions. In

the following two subsections we will outline how these differential equations can be solved and in the following section, we will discuss how the energy spectrum is obtained from the solutions.

3.4.1 Underlying space

In the underlying space, $\mathcal{U}$, the position and momentum operators, x and p, are defined by Eqs. (2.24) and (2.25) respectively. The SHO Hamiltonian takes the form

$$H = \frac{\hbar^2 \tan^2(\phi)}{2a^2 m} - \frac{1}{2}a^2 m\omega^2 \frac{\partial^2}{\partial \phi^2}, \tag{3.29}$$

and the corresponding eigenvalue problem becomes

$$\left(\frac{\hbar^2 \tan^2(\phi)}{2a^2 m} - E\right)\psi(\phi) - \frac{1}{2}a^2 m\omega^2 \frac{\partial^2 \psi(\phi)}{\partial \phi^2} = 0. \tag{3.30}$$

Making the substitution

$$\psi(\phi) \to \sqrt{\cos(\phi)}\psi(s)), \tag{3.31}$$

where $s = \sin(\phi)$, this differential equation becomes the associated Legendre equation in $\psi(s)$:

$$(1 - s^2)\frac{\partial^2 \psi(s)}{\partial s^2} - 2s\frac{\partial \psi(s)}{\partial s} + \left(N(N+1) - \frac{M^2}{1 - s^2}\right)\psi(s) = 0, \tag{3.32}$$

where two new dimensionless parameters, N and M, have been introduced:

$$N = \frac{\sqrt{2Ema^2 + \hbar^2}}{m\omega a^2} - \frac{1}{2} \tag{3.33}$$

$$M = \frac{\sqrt{4\hbar^2 + m^2\omega^2 a^4}}{2m\omega a^2}. \tag{3.34}$$

The associated Legendre functions, $P_{NM}(s)$ are not uncommon in physics; however, in this form they are a bit peculiar. Usually, the parameters of the function, in this case N and M, take on integer values, and the argument is the cosine of some

angle. In this case, the function becomes a polynomial in two variables, the cosine and the sine of the angle. In our case, the argument is the sine of the angle ϕ, which alone would not be unusual. However, M depends entirely on constants of nature and cannot be assumed to take an integer value. When either N or M are nonintegral, the associated Legendre functions, $P_{NM}(s)$. are singular when their argument is ± 1. This corresponds to $\phi = \pm\frac{\pi}{2}$, if we take $-\pi \leq \phi \leq \pi$. Thus we expect the wave function to have singularities on the domain under consideration.

In standard quantum mechanics, one finds the energy of the bound states by requiring that the wave function be normalizable. When solving the eigenvalue problem in position or momentum space, this requirement forces the wave function to approach zero sufficiently fast at $\pm\infty$. In our case, this requirement does not make sense, since our coordinate, ϕ, ranges over a finite interval. Furthermore, we cannot require that the wave function not have singularities, since M is nonintegral and therefore the associated Legendre function will always have singularities on the interval in question. Thus, we require that the singularities at $\phi = \pm\frac{\pi}{2}$ be mild enough to allow the wave function to be square integrable. We are unable to determine what restrictions on the energy will give a normalizable wave function. However, we can solve the eigenvalue problem in momentum space and determine the normalization condition and find the energy spectrum. Having found the energy spectrum, we can see what restriction must be placed on N to result in a normalizable wave function in the ϕ representation.

3.4.2 Momentum space

In momentum space, we adopt the representation given in Eqs. (2.49) and (2.50). The eigenvalue problem then becomes the differential equation

$$\left(\frac{p^2}{2m} - E\right)\psi(p) - \alpha m\omega^2\hbar^2 p\left(1+\alpha p^2\right)\frac{\partial\psi(p)}{\partial p} - \frac{1}{2}m\omega^2\hbar^2\left(1+\alpha p^2\right)^2\frac{\partial^2\psi(p)}{\partial p^2} = 0. \tag{3.35}$$

This equation also has a solution in terms of associated Legendre functions. Before giving the solution; however, we introduce the dimensionless parameters, ζ, β, and ϵ such that

$$\zeta = \frac{p}{\sqrt{m\hbar\omega}} \tag{3.36}$$

$$\beta = \alpha m\omega\hbar \tag{3.37}$$

$$\epsilon = \frac{E}{\hbar\omega}. \tag{3.38}$$

These parameters reduce the equation to

$$\left(\frac{\zeta^2}{2} - \epsilon\right)\psi(\zeta) - \beta\zeta\left(1+\beta\zeta^2\right)\frac{\partial\psi(\zeta)}{\partial\zeta} - \frac{1}{2}\left(1+\beta\zeta^2\right)^2\frac{\partial^2\psi(\zeta)}{\partial\zeta^2} = 0. \tag{3.39}$$

One can see that in the limit that β approaches zero, we recover the standard momentum space SHO eigenvalue problem. By introducing

$$N = \frac{\sqrt{4+\beta^2}}{2\beta} - \frac{1}{2} \tag{3.40}$$

$$M = \frac{\sqrt{1+2\,\beta\epsilon}}{\beta} \tag{3.41}$$

and letting

$$\psi(\zeta) \to \psi(i\sqrt{\beta}\zeta) \tag{3.42}$$

this equation reduces to the associated Legendre equation (3.32) with N and M as parameters. It should be noted that N and M as defined in Eqs. (3.40) and (3.41) differ from the N and M defined in Eqs. (3.33) and (3.34). In both cases, N and M

are simply the parameters that appear in the associated Legendre function, although the parameters differ depending on which representation is employed. In particular, in the underlying space the energy appears in N, while in the momentum representation it appears in M.

As in the underlying space, the quantization of energy results from the requirement that the wave function be normalizable. In momentum space, the argument of the associated Legendre functions vary over the pure imaginary numbers, and so the wave function has no singularities for any value of momentum, except possibly at $\pm\infty$. Thus the normalization condition requires that the wave function approach zero sufficiently fast as p approaches $\pm\infty$ so that the resulting function be square integrable. The exact form of this condition is not obvious and will be discussed in the next section, 3.5.

As we commented in section 3.2, the square modulus of the wave function does not represent the probability density because the inner product between two vectors includes an extra weight factor in the volume element. Instead, the probability density is proportional to the square modulus of the expression

$$\frac{\psi(\zeta)}{\sqrt{1+\beta\zeta^2}} = \frac{P_{NM}(i\sqrt{\beta}\zeta)}{\sqrt{1+\beta\zeta^2}}. \tag{3.43}$$

In section 3.2, we suggested that the position operator representation

$$x = i\hbar\left(\frac{\partial}{\partial p} + \alpha p\frac{\partial}{\partial p}p\right), \tag{3.44}$$

leads to a wave function that naturally includes this weight factor. In the case of the SHO, this can be verified. Solving the eigenvalue differential equation in a manner similar to the one outlined above, results in the wave function

$$\psi(\zeta) = \frac{P_{NM}(i\sqrt{\beta}\zeta)}{\sqrt{1+\beta\zeta^2}}. \tag{3.45}$$

Furthermore, if we define the position operator to be

$$x = i\hbar \frac{\partial}{\partial p}\left(1 + \alpha p^2\right), \tag{3.46}$$

then the resulting wave functions take the form

$$\psi(\zeta) = \frac{P_{NM}(i\sqrt{\beta}\zeta)}{1+\beta\zeta^2}, \tag{3.47}$$

which is consistent with the hypothesis that taking different operator transformations results in the same energy spectrum, but with modified wave function that can be related to the probability density.

3.5 Energy spectrum

To find the energy spectrum of the SHO in Snyder space, we look at the asymptotic behavior of the associated Legendre functions as they appear in the solution of the momentum space representation. It has been shown that if N and M are the parameters of the function and if N and M do not differ by a positive integer, then the function does not vanish as p approaches $\pm\infty$ [20]. Therefore, the only possible normalizable functions must satisfy

$$M - N = n + 1, \tag{3.48}$$

where n is a non-negative integer. Applying this condition and solving for the energy gives

$$E_n = \hbar\omega\epsilon_n = \hbar\omega\left(\sqrt{1+\left(\frac{\beta}{2}\right)^2}\left(n+\frac{1}{2}\right) + \frac{\beta}{2}\left(\left(n+\frac{1}{2}\right)^2 + \frac{1}{4}\right)\right). \tag{3.49}$$

Although we have not shown explicitly that the condition presented in Eq. (3.48) results in a square integrable wave function, we have shown that every other possibility results in a function that is not square integrable. Thus Eq. (3.48) includes all

the energy levels; however, it is possible that it includes some energies that are not valid solutions. One reason to suspect that this is not the case and that Eq. (3.48) represents the complete and correct spectrum, is that in the limit that β (and therefore a) goes to zero, we recover the energy spectrum of the SHO in standard quantum mechanics. The analyticity of the perturbation in α, and likewise β, guarantees that all n values in Eq. (3.49) are valid in order to recover all energy levels of the SHO in standard quantum mechanics.

If we apply the definition of the energy spectrum in Eq. (3.49) to the definition of N in the underlying space, Eqs. (3.33) and (3.34), we find that the normalization condition forces N and M to differ by a non-negative integer. This is very similar to the condition in momentum space but allows for the possibility of $N = M$.

It should be noted that the SHO with minimal length uncertainty relations, which includes Snyder space as a special case, has been solved exactly using a very different technique from the one we present in this chapter, but leading to the same energy spectrum [12]. We feel that Eq. (3.49) is correct, and the agreement with the result from the literature validates our method.

Finally, we compare the energy level and wave functions of Snyder space and standard quantum mechanics. We plot several momentum space wave functions and energy levels for various values of β and n in Figs. 3.1 to 3.6. It should be noted that the wave function as described in Eq. (3.45) is generally complex; however, as in standard quantum mechanics the phase is constant, so one may arbitrarily choose the phase to be zero as we have done to generate these figures.

For each of the states in Figs. 3.1 to 3.6, the solution for $\beta = .01$ agrees with standard quantum mechanics, i.e. $\beta = 0$, very closely, as we expect. For larger values of β we see that the relative amplitude of the wave function grows near the origin. Furthermore, the peaks of the wave function become narrower with increasing

β. Thus, it appears that Snyder space "squeezes" the momentum towards zero. "Squeezing" the momentum towards zero, corresponds to spreading the spatial part of the wave function away from zero. Qualitatively, this means that the particle spends more time in areas of high potential energy. It is has not been shown whether the virial theorem holds in Snyder space.

Qualitatively we can understand the "squeezing" of the momentum towards zero by considering the effects of quantized spacetime. In standard quantum mechanics, the position space eigenfunction of the n^{th} energy level has n nodes. In Snyder space, we cannot discuss nodes because there is no continuous wave function in position space. However, in Snyder space, we have a discrete position space representation of the wave function as we saw in chapter 2. We generalize the notion of a node to be when two of the coefficient of two successive position eigenstates change sign. Since all nodes occur in the classically allowed region, we can consider the average length between nodes. In standard quantum mechanics, as $n \to \infty$, this average distance between nodes approaches zero, since the energy is linear in n, while the size of the classically allowed region is quadratic. In Snyder space, however, both the total energy and the classically allowed region are quadratic. In the limit that $n \to \infty$, the spacing between nodes approaches $a/2$. This result agrees qualitatively with what we expect from quantized spacetime since there cannot be more sign changes than there are coefficients of position eigenfunctions.

3.6 Size of perturbation

The measureability of the results outlined in this chapter depends on the magnitude of the deviation from the standard theory. Naturally the size of the perturbation depends on the size of the fundamental unit of length, a. Since currently there is no

Figure 3.1 One dimensional SHO ground state wave functions

Figure 3.2 One dimensional SHO first excited state wave functions

Figure 3.3 One dimensional SHO second excited state wave functions

Figure 3.4 One dimensional SHO third excited state wave functions

Figure 3.5 One dimensional SHO fourth excited state wave functions

Figure 3.6 One dimensional SHO fifth excited state wave functions

known discrepancy between experiment and the standard theory, we can only place an upper bound on the value of a.

In theories of quantum gravity, the Planck length is usually, but not universally [6], accepted as the length at which gravitational effects will be significant.

The Planck length can be calculated by combining three fundamental constants, the speed of light, c, the gravitational constant G, and Planck's constant $\hbar$, so as to produce a constant with dimensions of length. It turns out that there is a unique solution:

$$l_p = \sqrt{\frac{G\hbar}{c^3}}, \tag{3.50}$$

which is approximately 10^{-35} meters.

While it is generally accepted that the fundamental length must be on the order of the Planck length, not everyone agrees. In particular, Meschini argues that if a theory of quantum gravity introduces new fundamental constants, such as Snyder's a, then the reasoning from which the value for the Planck length was derived breaks down, since there is a greater number of fundamental constants than the three that Planck considered [6]. By analogy, the action scale at which quantum effects become apparent, $\hbar$, could not have been derived by combining classical constants. Snyder's fundamental length a may be as irreducible as Planck 's constant, $\hbar$.

If the actual value of a is to be determined experimentally, it must be much larger than the Planck length. In a publication regarding the minimal-length uncertainty relations, Chang *et al.* estimate that current experiments could resolve a fundamental length on the order of 10^{-19} meters [12]. This is is about 16 orders of magnitude larger than the Planck length. Thus, even if the Planck length grossly underestimates the actual value of a, it is unlikely that an experimental verification will be available soon. However, if a is independent of the Planck length, as Meschini argues, then experimental verification could be feasible.

Since the energy spectrum of the SHO in Snyder go as n^2 while the spectrum from standard quantum mechanics goes as n, for sufficiently high energies there will always be a significant deviation from the standard theory. If there is going to be an experimental test of the results in this chapter, it will most likely employ this characteristic.

3.7 Conclusions

We have found an exact solution of the SHO energy eigenvalue problem: we give both the new energy spectrum in Eq. (3.49) and the momentum space wave functions in Eq. (3.45). The deviations from the standard result are minimal for a small fundamental length, and it is currently not possible to verify the results.

In addition to the closed form solution of the one-dimensional SHO, the techniques presented in this chapter could be used to study other systems in Snyder space. The most general result from this chapter is the methodology for approaching eigenvalue problems in Snyder space. We employed two methods to solve the SHO eigenvalue problem.

The first method used demonstrates that Snyder space can be formulated as a perturbation of standard quantum mechanics. Operator transformations can be used to relate the position and momentum operators, x_S and p_S of Snyder space to operators analogous to the position and momentum operators of standard quantum mechanics, x and p. Choosing a particular operator transformation is related to hermiticity, although we conjecture that the energy spectrum is independent of the transformation used.

The second method shows that differential equations can be formulated to find solutions in Snyder space. The explicit analytic solutions for the one-dimensional

SHO case also establishes an existence proof for soluble polynomial systems in Snyder space.

Having studied the one-dimensional SHO eigenvalue problem, we turn our attention to the two-dimensional SHO in the next chapter in which we seek to generalize the results and methods of this chapter to higher dimensions. We will see that the methods can be equally successfully applied in two dimensions.

Chapter 4

Two-Dimensional Simple Harmonic Oscillator

4.1 Operator transformations

Having studied the one-dimensional harmonic oscillator, we turn our attention to the two-dimensional isotropic harmonic oscillator. In standard quantum mechanics, the energy spectrum of the two-dimensional isotropic SHO is simply the sum of two one-dimensional SHOs. In Snyder space we find that this is not the case, as the results in this chapter demonstrate. This result is a consequence of the fact that two-dimensional standard quantum mechanics is essentially the tensor product of two systems in one dimension. In Snyder space this is not the case, as can be seen from the nonvanishing commutator between the two position operators. Our approach to solving the energy eigenvalue problem parallels that of chapter 3. We calculate a perturbation to the energy spectrum of the SHO in standard quantum mechanics. We also formulate the eigenvalue problem as a differential equation.

In two dimensions, the commutation relations in Eq. (3.1), get augmented to

$$[x, p_x] = i\hbar\left(1 + \alpha p_x^2\right) \tag{4.1}$$

$$[y, p_y] = i\hbar\left(1 + \alpha p_y^2\right) \tag{4.2}$$

$$[x, p_y] = [y, p_x] = i\hbar\alpha p_x p_y \tag{4.3}$$

$$[x, y] = i\hbar\alpha L_z \tag{4.4}$$

$$[p_x, p_y] = 0 \tag{4.5}$$

$$[x, L_z] = -i\hbar y \tag{4.6}$$

$$[y, L_z] = i\hbar x \tag{4.7}$$

$$[p_x, L_z] = -i\hbar p_y \tag{4.8}$$

$$[p_y, L_z] = i\hbar p_x. \tag{4.9}$$

If we note that L_z is related to the position and momentum operators by

$$L_z = xp_y - yp_x, \tag{4.10}$$

then Eqs. (4.6) - (4.9) are automatically satisfied and the number of commutation relations reduces from 10 to 6. In either case, this commutation algebra is significantly more complicated than the single commutation relation in one dimension. The Hamiltonian for the isotropic harmonic oscillator in two dimensions is defined as in standard quantum mechanics to depend on the radial variable r only

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2\left(x^2 + y^2\right) = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2 r^2. \tag{4.11}$$

We will proceed as in the one-dimensional case and let x_S, y_S, p_{Sx}, and p_{Sy} represent the position and momentum operators of Snyder space obeying Eqs. (4.1) - (4.5), and let x, y, p_x, and p_y represent operators satisfying the canonical commutation relations

$$[x, p_x] = [y, p_y] = i\hbar, \tag{4.12}$$

with all other commutators vanishing. We search for solutions to the the equations

$$x_S = x_S(x, p_x, y, p_y) \tag{4.13}$$

$$y_S = y_S(x, p_x, y, p_y) \tag{4.14}$$

$$p_{Sx} = p_{Sx}(x, p_x, y, p_y) \tag{4.15}$$

$$p_{Sy} = p_{Sy}(x, p_x, y, p_y). \tag{4.16}$$

It turns out that there are many solutions to these equations. If we take

$$p_{Sx} = p_x \tag{4.17}$$

$$p_{Sy} = p_y \tag{4.18}$$

$$x_S = x + \alpha\left(p_x^2\, x\, t + x\, p_x^2\,(1-t) + p_x\,(y\, p_y\, u + p_y\, y\,(1-u))\right) \tag{4.19}$$

$$y_S = y + \alpha\left(p_y^2\, y\, v + y\, p_y^2\,(1-v) + p_y\,(x\, p_x\, w + p_x\, x\,(1-w))\right) \tag{4.20}$$

where $0 \leq t, u, v, w \leq 1$, then all the commutation relations are satisfied except the one between x_S and y_S. It can be shown that $[x, y] = i\hbar\alpha L_z$ if

$$2t - 2v + w - u = 0. \tag{4.21}$$

Thus there are an infinite number of transformations to choose from. This is analogous to the one-dimensional case where we found that the transformation given by Eq. (3.6) where $0 \leq t \leq 1$ satisfies $[x_S, p_{Sx}] = i\hbar\,(1 + \alpha p_{Sx}^2)$.

As in the one-dimensional case, we focus on the specific cases of when t, u, v, and w each are 1, 1/2, and 0, then there are a total of 81 possible transformations given in Eqs. (4.17) - (4.20). However, it can be shown that only 13 of these transformations satisfy $[x, y] = i\hbar\alpha L_z$. We choose these cases for the same reasons described in section 3.2. If we associate x and y with differentiation with respect to p_x and p_y respectively, then the case in which $t = u = v = w = 1$ corresponds to the momentum space solution given by Snyder in Eq. (2.54) and (2.55). Furthermore, we conjecture that

the case in which $t = u = v = w = 1/2$ will result in a wave function that naturally includes the weight factor from the volume element. The case for $t = u = v = w = 0$ is the opposite extreme in normal ordering. We consider all other possible combinations in which t, u, v, and w are each 1, 1/2, or 0 for completeness.

Not all possible transformations are given by simple polynomial expressions as in Eqs. (4.17) - (4.20). In addition, it can be shown that

$$x_S = x \tag{4.22}$$

$$p_{Sx} = \frac{1}{\sqrt{\alpha}} \tan\left(\sqrt{\alpha} p_x\right) \tag{4.23}$$

$$y_S = -y \cos\left(\sqrt{\alpha} p_x\right) + \cot\left(\sqrt{\alpha} p_y\right) sin\left(\sqrt{\alpha} p_x\right) x \tag{4.24}$$

$$p_{Sy} = \frac{1}{\sqrt{\alpha}} \frac{\cot\left(\sqrt{\alpha} p_y\right)}{\cos\left(\sqrt{\alpha} p_x\right)} \tag{4.25}$$

also satisfy the commutation relations in Eqs. (4.1) - (4.5).

As in the one-dimensional case, we will proceed by assuming that all transformations will result in the same energy spectrum and that the wave functions will differ by some simple multiplicative factor.

In order to apply perturbation theory, as in the one dimensional case, it will be necessary for x, y, p_x, and p_y each be Hermitian in order for the ladder operators to have their usual behavior. Therefore, we will be forced to use either the transformation corresponding $t = u = v = w = 1/2$ or the transformation in Eq. (4.22) - (4.25)

4.2 Complexity with increasing dimensions

As we saw in the previous section, the complexity between the commutation algebras increases from one to two dimensions. In one dimension there was only a single commutation relation. In two dimensions, we found that there were 10 commutators,

although four were redundant because they involved the angular momentum operator which is defined in terms of position and momentum. With each dimension, there are two additional operators, the position and momentum operators. Therefore, for d dimensions, there will be $\begin{pmatrix} 2d \\ 2 \end{pmatrix}$ commutation relations in the commutation algebra. Thus the number increases to 15 commutators for three dimensions and 28 for four dimensions.

The number of commutators that define the algebra for arbitrary dimensions is the same in Snyder space as in standard quantum mechanics; however, in standard quantum mechanics, the majority of those commutation relations are zero, while in Snyder space, the only vanishing commutation relations are among the momenta. The number of nonzero commutators is the major source of complexity in Snyder space. In particular, the nonzero commutator of the position operators causes difficulty both conceptually and mathematically.

An added complication of multiple dimensions in Snyder space is that if one is looking for operator transformations analogous to Eqs. (4.17) - (4.20), then each new dimension adds an additional term to x_S, y_S, etc. Furthermore, if we restrict our attention to the transformations in which the coefficients of each of those terms are 0, 1/2, and 1 for the reasons described above, then there are three possible coefficients for each term, and there is one additional term for each dimension, and one additional operator for each dimension. Thus there are a total of 3^{d^2} possible transformations. Of these transformations, it is not guaranteed that they will satisfy all of the commutation relations. In particular, in two dimensions we saw that of the 81 possibilities, only 13 satisfied the commutation relation between x and y. In three dimensions there are 19,683 possibilities of which 399 satisfy the the commutation relations among the position operators as can be checked using the algorithms in

Appendix B.1.3.

Since it has not been formally shown that all transformations result in the same energy spectrum, the choice of transformations becomes important. In particular, the relative simplicity of a single dimension could have been the source of the coincidental equality of the energy spectra. In the next section we actually calculate the perturbed energy spectra for various transformations to address this question.

4.3 Perturbation treatment

In two dimensions we choose an operator transformation and express the resulting Hamiltonian in terms of raising and lowering operators, as in the one-dimensional case. Choosing the transformation in which all the coefficients are 1/2, the resulting Hamiltonian is

$$H = H_0 + V_\alpha + V_{\alpha^2} \tag{4.26}$$

$$H_0 = \omega\,\hbar\left(a^\dagger a + b^\dagger b + 1\right) \tag{4.27}$$

$$\begin{aligned} V_\alpha = {} & \frac{1}{4}\,\omega^2 h^2 \alpha\, m\left(2\,{b^\dagger}^2 b^2 + 4\,b^\dagger b + 2 + 4\,a^\dagger a + 2\,{a^\dagger}^2 a^2\right) \\ & + \frac{1}{4}\,\omega^2 h^2 \alpha\, m\left(-{b^\dagger}^4 - 2\,a^2 b^2 - b^4 + 2\,a^2 {b^\dagger}^2\right) \\ & + \frac{1}{4}\,\omega^2 h^2 \alpha\, m\left(2\,{a^\dagger}^2 b^2 - 2\,{a^\dagger}^2 {b^\dagger}^2 - {a^\dagger}^4 - a^4\right) \end{aligned} \tag{4.28}$$

$$\begin{aligned} V_{\alpha^2} = {} & \frac{1}{16}\omega^3 \hbar^3 m^2 \alpha^2\left(18 + 20\,{b^\dagger}^2 b^2 + 16\,a^\dagger a b^\dagger b + 34\,b^\dagger b + 4\,{a^\dagger}^3 a^3 + 4\,{a^\dagger}^2 a^2 {b^\dagger}^2 b^\dagger b\right) \\ & + \frac{1}{16}\omega^3 \hbar^3 m^2 \alpha^2\left(4\,{b^\dagger}^3 b^3 + 4\,a^\dagger a {b^\dagger}^2 b^2 + 34\,a^\dagger a + 20\,{a^\dagger}^2 a^2\right) \\ & + \frac{1}{16}\omega^3 \hbar^3 m^2 \alpha^2\left(-9\,a^2 - 9\,b^2 - 9\,{a^\dagger}^2 - 9\,{b^\dagger}^2\right) \\ & + \frac{1}{16}\omega^3 \hbar^3 m^2 \alpha^2\left(-6\,{a^\dagger}^4 - 4\,{a^\dagger}^2 b^\dagger b - 4\,b^\dagger b^3 + 12\,{a^\dagger}^2 b^2 - 12\,{a^\dagger}^2 {b^\dagger}^2\right) \\ & + \frac{1}{16}\omega^3 \hbar^3 m^2 \alpha^2\left(12\,a^2 {b^\dagger}^2 - 4\,a^\dagger a b^2 - 4\,{b^\dagger}^3 b - 4\,a^\dagger a^3 - 6\,a^4 - 6\,{b^\dagger}^4\right) \\ & + \frac{1}{16}\omega^3 \hbar^3 m^2 \alpha^2\left(-6\,b^4 - 4\,a^2 b^\dagger b - 4\,{a^\dagger}^3 a - 4\,a^\dagger a {b^\dagger}^2 - 12\,a^2 b^2\right) \end{aligned} \tag{4.29}$$

$$
\begin{aligned}
&+\frac{1}{16}\omega^3\hbar^3m^2\alpha^2\left(-2\,a^{\dagger 4}b^\dagger b+4\,a^2b^{\dagger 3}b-a^{\dagger 4}b^2+3\,a^{\dagger 4}b^{\dagger 2}+3\,a^4b^2-a^4b^{\dagger 2}\right)\\
&+\frac{1}{16}\omega^3\hbar^3m^2\alpha^2\left(4\,a^{\dagger 3}ab^2-4\,a^{\dagger 3}ab^{\dagger 2}-4\,a^\dagger a^3b^2+4\,a^\dagger a^3b^{\dagger 2}-a^{\dagger 4}a^2\right)\\
&+\frac{1}{16}\omega^3\hbar^3m^2\alpha^2\left(-a^{\dagger 2}a^4-2\,b^{\dagger 5}b-b^{\dagger 4}b^2-2\,a^{\dagger 2}a^2b^2-2\,a^2b^{\dagger 2}b^2+a^{\dagger 6}\right)\\
&+\frac{1}{16}\omega^3\hbar^3m^2\alpha^2\left(-2\,a^{\dagger 5}a-b^{\dagger 2}b^4-2\,a^4b^\dagger b-4\,a^{\dagger 2}b^{\dagger 3}b-2\,a^\dagger ab^{\dagger 4}-a^{\dagger 2}b^4\right)\\
&+\frac{1}{16}\omega^3\hbar^3m^2\alpha^2\left(-2\,a^{\dagger 2}b^{\dagger 2}b^2+b^{\dagger 6}+a^6+4\,a^{\dagger 2}b^\dagger b^3-2\,a^\dagger ab^4+3\,a^2b^4\right)\\
&+\frac{1}{16}\omega^3\hbar^3m^2\alpha^2\left(-2\,a^{\dagger 2}a^2b^{\dagger 2}+b^6+3\,a^{\dagger 2}b^{\dagger 4}-2\,b^\dagger b^5\right)\\
&+\frac{1}{16}\omega^3\hbar^3m^2\alpha^2\left(-4\,a^2b^\dagger b^3-a^2b^{\dagger 4}-2\,a^\dagger a^5\right)
\end{aligned}
$$

where a and $a^\dagger$ are defined in Eqs. (3.8) and (3.9) and

$$b=\sqrt{\frac{m\omega}{2\hbar}}\left(y+\frac{i}{m\omega}p_y\right) \tag{4.30}$$

$$b^\dagger=\sqrt{\frac{m\omega}{2\hbar}}\left(y-\frac{i}{m\omega}p_y\right). \tag{4.31}$$

One can show that a, $a^\dagger$, b, and $b^\dagger$ satisfy the commutation algebra

$$[a,a^\dagger]=1 \tag{4.32}$$

$$[b,b^\dagger]=1 \tag{4.33}$$

$$[a,b]=[a,b^\dagger]=[b,a^\dagger]=[a^\dagger,b^\dagger]=0. \tag{4.34}$$

The structure of the Hamiltonian is similar to that of the one-dimensional case. The first term is the unperturbed Hamiltonian. There are also terms proportional to α and to α^2, although they are much more complicated in two-dimensions.

In two dimensions the isotropic SHO has a degenerate spectrum. Degenerate perturbation theory needs to be used and is discussed briefly in appendix C. To guarantee that the perturbation converges uniformly, the correct basis of eigenvectors of the unperturbed Hamiltonian must be used in evaluating the corrections.

In two dimensions we have the same situation as in one dimension; the perturbation has terms proportional to both α and α^2. Because of the complexity of the

α^2 term, we calculate the perturbation correction to second order. Thus, only V_α needs to be included in the second order perturbation term for which we only need to diagonalize the matrix $V_{\alpha\ mn} = \langle m|V_\alpha|n\rangle$. However, in order to include all second order corrections, we still include V_{α^2} in the first order perturbation correction.

The calculation of the matrix elements and diagonalization of the corresponding matrix is straightforward but lengthy, as it has to be done for each degenerate energy level. To find the correct basis for perturbation theory, we developed a Maple algorithm that significantly reduces the complexity of the computation.

It turns out that for each energy level, diagonalizing the matrix does not completely remove the degeneracy. In particular, an even number of originally degenerate eigenstates of the unperturbed Hamiltonian leads to a new basis with twofold degeneracy. An odd number of originally degenerate eigenstates of the unperturbed Hamiltonian leads to only one nondegenerate eigenstate while other basis states exhibit twofold degeneracy. The nondegenerate basis state is always the highest energy state within the degenerate subspace.

As we will see in the next section, the energy levels of the Snyder space SHO are either doubly degenerate or non-degenerate according to the pattern described in the previous paragraph. Thus, the perturbation series will converge uniformly.

The actual energy shifts were found using the Maple algorithm found in appendix B.3.2. The energy shifts for the first few energy levels are given below:

$$E_{0,1} = \hbar\omega\left(1 + \frac{1}{2}\beta + \frac{1}{8}\beta^2\right) \tag{4.35}$$

$$E_{1,1} = \hbar\omega\left(2 + \frac{3}{2}\beta + \frac{1}{4}\beta^2\right) \tag{4.36}$$

$$E_{1,2} = \hbar\omega\left(2 + \frac{3}{2}\beta + \frac{1}{4}\beta^2\right) \tag{4.37}$$

$$E_{2,1} = \hbar\omega\left(3 + \frac{5}{2}\beta + \frac{3}{8}\beta^2\right) \tag{4.38}$$

$$E_{2,2} = \hbar\omega\left(3 + \frac{5}{2}\beta + \frac{3}{8}\beta^2\right) \tag{4.39}$$

$$E_{2,3} = \hbar\omega\left(3 + \frac{9}{2}\beta + \frac{1}{2}\beta^2\right) \tag{4.40}$$

$$E_{3,1} = \hbar\omega\left(4 + \frac{7}{2}\beta + \frac{1}{2}\beta^2\right) \tag{4.41}$$

$$E_{3,2} = \hbar\omega\left(4 + \frac{7}{2}\beta + \frac{1}{2}\beta^2\right) \tag{4.42}$$

$$E_{3,3} = \hbar\omega\left(4 + \frac{15}{2}\beta + \frac{1}{2}\beta^2\right) \tag{4.43}$$

$$E_{3,4} = \hbar\omega\left(4 + \frac{15}{2}\beta + \frac{1}{2}\beta^2\right), \tag{4.44}$$

where $E_{n,m}$ is meant to represent the n^{th} unperturbed energy level and its m^{th} degeneracy.

We now proceed to identify patterns in the corrections. The first order correction is always of the form $\frac{\beta}{2}((n+1)^2 - l^2)$, where n is the principal quantum number of the unperturbed energy and l is an integer taking values n, $n-2$, ..., $-n$. We have suggestively called the second quantum number l. At this point we have not shown that l is related to angular momentum; however, we know that the angular momentum is a perfectly good quantum number since it commutes with the Hamiltonian. We show in the next section, that l is in fact the angular momentum quantum number. The second-order terms are of the form $\frac{\beta^2}{8}(n+1)$. These two observations inspire the conjecture that in general,

$$E_{n,l} = \hbar\omega\left(n + 1 + \frac{\beta}{2}\left((n+1)^2 - l^2\right) + \frac{\beta^2}{8}(n+1)\right). \tag{4.45}$$

The similarity between this energy spectrum and the one-dimensional spectrum Eq. (3.28) strengthens the conjecture. Extending the analogy with one dimension, we could further conjecture that the exact form of the energy is of the form

$$E_{n,l} = \hbar\omega\left((n+1)\sqrt{1 + \left(\frac{\beta}{2}\right)^2} + \frac{\beta}{2}\left((n+1)^2 - l^2\right)\right). \tag{4.46}$$

In the next section we see that this conjecture is correct, as well.

4.4 Exact treatment

As in the one-dimensional case, we attempt to solve the energy eigenvalue problem in the underlying space. We begin by writing the Hamiltonian in the form

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2\left(x^2 + y^2\right) = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2\left(r^2 + \alpha L_z^2 - \alpha L_z^2\right). \tag{4.47}$$

We have added $0 = \alpha L_z^2 - \alpha L_z^2$ to the Hamiltonian in order to take advantage of the result found in Chapter 2:

$$r^2 + \alpha L_z^2 = -a^2\left(\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \tag{4.48}$$

$$\frac{p_x^2 + p_y^2}{2m} = \frac{\hbar^2\tan^2(\theta)}{2ma^2} \tag{4.49}$$

$$L_z^2 = -\hbar^2\frac{\partial^2}{\partial\phi^2}. \tag{4.50}$$

Furthermore, we saw in chapter 2 that any radially symmetric Hamiltonian has common eigenstates with the angular momentum operator. Therefore, we assume a solution of the energy eigenvalue problem of the form

$$\langle\phi,\theta|\psi\rangle \ = f(\theta)e^{il\phi}, \tag{4.51}$$

where $\hbar l$ is the eigenvalue of L_z. Applying these definitions to the energy eigenvalue problem

$$\langle\phi,\theta|H|\psi\rangle \ = E\langle\phi,\theta|\psi\rangle, \tag{4.52}$$

we find the following differential equation for $f(\theta)$:

$$\left(\frac{\tan^2(\theta)}{2m\alpha} - E + \frac{1}{2}m\alpha\hbar^2\omega^2 l^2\cot^2(\theta)\right)f(\theta) - \alpha m\omega^2\hbar^2\left(\frac{1}{\sin(\theta)}\frac{d}{d\theta}\sin(\theta)\frac{d\ f(\theta)}{d\theta}\right) = 0. \tag{4.53}$$

We introduce the dimensionless parameters β and ϵ

$$\beta = \alpha m\omega\hbar \tag{4.54}$$

$$\epsilon = \frac{E}{\hbar\omega} \tag{4.55}$$

and make a change of variables

$$P = tan(\theta), \tag{4.56}$$

under which the equation becomes:

$$\left(\frac{P^2}{2\beta} - \epsilon - \frac{\beta l^2}{2P^2}\right) f(P) + \frac{\beta}{P}\left(1+P^2\right)\left(1+2P^2\right) f'(P) + \beta(1+P^2)^2 f''(P) = 0. \tag{4.57}$$

This equation has an exact solution in terms of the hypergeometric function; however, it is unclear from the solution what restriction must be placed on the energy to result in a normalizable wave function. One could, in principle, calculate the energy values using numerical methods, but that technique will not be attempted here.

If the eigenvalue problem is formulated in momentum space, in which x and y take the form given in Eqs. (2.54) and (2.55), the resulting partial differential equation is not separable. However, upon making the change of variables

$$P^2 = \alpha\left(p_x^2 + p_y^2\right) \tag{4.58}$$

$$\Phi = \tan\left(\frac{p_y}{p_x}\right), \tag{4.59}$$

all Φ dependence drops out and the equation reduces to Eq. (4.57). This should be expected since

$$P^2 = \alpha\left(p_x^2 + p_y^2\right) = tan^2(\theta) \tag{4.60}$$

in both representations.

4.5 Energy spectrum

As mentioned in the previous section, it is unclear from the solution of the differential equation what values of the energy will result in a normalizable wave function and hence give the exact energy spectrum. However, we have calculated the second order perturbation corrections to the energy in Eq. (4.45). The form of this perturbation

is very similar to the one-dimensional perturbation. This inspires the conjecture that the exact spectrum is

$$E_n = \hbar\omega \left((n+1)\sqrt{1+\left(\frac{\beta}{2}\right)^2} + \frac{\beta}{2}\left((n+1)^2 - l^2\right) \right). \tag{4.61}$$

It is unclear how this energy spectrum can be extracted from the solution obtained in the previous section.

As mentioned in chapter 1, Chang *et al.* [12] have solved the SHO spectrum exactly for the minimal length uncertainty relations in arbitrary dimensions. Since Snyder space is a special case of the minimal length uncertainty algebra, it can be shown using the result in the literature that the conjecture given above is correct. I discuss the relation between Snyder space and the minimal length uncertainty relations further in section 5.4.

4.6 Anisotropy of the fundamental length considerations

An interesting question to consider in multiple dimensions is how the spectrum changes when the fundamental length is not the same in both dimensions. If we alter the definition of the position operators in Snyder space to be

$$x' = ia\left(\eta_4\frac{\partial}{\partial\eta_1} - \eta_1\frac{\partial}{\partial\eta_4}\right) \tag{4.62}$$

$$y' = ib\left(\eta_4\frac{\partial}{\partial\eta_2} - \eta_2\frac{\partial}{\partial\eta_4}\right), \tag{4.63}$$

and redefine the momentum operators as

$$p'_x = \frac{\hbar}{a}\frac{\eta_1}{\eta_4} \tag{4.64}$$

$$p'_y = \frac{\hbar}{b}\frac{\eta_2}{\eta_4}, \tag{4.65}$$

where $a \neq b$, then the fundamental length is anisotropic. I include a $'$ in the definition to distinguish the anisotropic operators from the isotropic operators used throughout the rest of this work. Introducing an anisotropic fundamental length destroys Lorentz invariance, which is one of the appealing characteristics of Snyder space.

In practice, we can relate the anisotropic operators to the usual ones by

$$x' = x \tag{4.66}$$

$$y' = \frac{a}{b}y \tag{4.67}$$

$$p'_x = p_x \tag{4.68}$$

$$p'_y = \frac{a}{b}p_y. \tag{4.69}$$

Using this relation, the Hamiltonian of the two-dimensional isotropic SHO becomes

$$H' = \frac{p'^2_x + p'^2_y}{2m} + \frac{1}{2}m\omega^2\left(x'^2 + y'^2\right) = \frac{p_x^2 + \frac{a^2}{b^2}p_y^2}{2m} + \frac{1}{2}m\omega^2\left(x^2 + \frac{a^2}{b^2}y^2\right). \tag{4.70}$$

This transformation relates the Hamiltonian of the anisotropic Snyder space to a similar problem in normal Snyder space. All the techniques discussed in the previous sections can be used to solve the eigenvalue problem of this new problem. The corrections to the energy spectrum can be calculated in anisotropic Snyder space.

4.7 Conclusions

The methods we have used to study the two-dimensional SHO are very similar to those used in chapter 3 for the one-dimensional SHO. In particular, we have used operator transformations to express the Snyder SHO as a perturbed SHO in standard quantum mechanics and have employed degenerate perturbation theory to obtain the second-order energy shifts. We have also expressed the energy eigenvalue problem as an exact differential equation. While the resulting differential equation has an exact

solution in terms of the hypergeometric function, it is not immediately apparent what quantization condition on the energy should be imposed. In practice, numerical techniques could be used to calculate the energy levels.

We have seen that increasing the number of dimensions leads to increasing complexity in the differential equations and operator transformations. While perturbation calculations are much more lengthy, the methods are straightforward and can be automated. I have been able to interpret the results thus obtained into conjectures for exact spectral formulas. These conjectures have been validated elsewhere in the literature. Thus, in principle the techniques used in the last two chapters could be applied to systems in arbitrary dimensions in Snyder space. The algorithms developed in this chapter show the way for perturbative calculations in higher-dimensional Snyder space.

Chapter 5

Snyder Space and Noncommutative Quantum Mechanics

5.1 Classification of non-commutative spaces

Recently there has been an enormous effort in the literature to study the effects of deforming the Heisenberg algebra of standard quantum mechanics, see for example [9, 17]. The field of noncommutative quantum mechanics, which is described below, encompasses much of the present effort to document these changes. It is interesting to note that many contributors cite Snyder's work as the historical precedence for doing quantum mechanics on a noncommutative manifold; however, very little work has actually been done in studying Snyder space. Figure 5.1 shows the number of papers citing Snyder's 1947 paper each year from 1984 to 2005 as given on the ISI Web of Science website in March 2006. (The ISI website only compiles bibliographic information through 1984.) It is curious that Snyder space is cited so often while little or no work has been done to study Snyder space in any detail. This work has been an effort to remedy this shortcoming by seriously investigating the consequences of

Figure 5.1 Citations of Snyder's 1947 original paper.

Snyder space. This chapter aims at putting Snyder space in its proper context within the modern study of noncommutative quantum mechanics.

As mentioned in the introductory chapter of this thesis, the Heisenberg algebra as defined in Eq. (1.2) - (1.4) is one of the fundamental ingredients of standard quantum mechanics from which all the properties of position and momentum can be derived. The interest in studying deformations of this algebra originates in the low energy limit of string theory, which predicts that Eq. (1.3) be modified to

$$[x_i, x_j] = i\Theta_{ij}, \tag{5.1}$$

where Θ_{ij} is a real antisymmetric matrix with dimensions of length squared. If the noncommutativity between different spatial dimensions is a consequence of some sort of quantization of spacetime, then the value of the fundametal length is estimated to be approximately $\sqrt{|\Theta_{ij}|}$. A study of this modified commutation algebra produced a number of interesting results discussed below [17].

Soon after this prediction of string theory, physicists began to study other defor-

mations of the Heisenberg algebra. A common deformation is obtained be letting Eq. (1.4) become

$$[p_i, p_j] = i\tilde{\Theta}_{ij}. \tag{5.2}$$

These two modifications are the simplest deformations of the Heisenberg algebra that can be considered and their study constitutes the field of noncommutative quantum mechanics. These two deformations are obtained by letting the vanishing commutators of the Heisenberg algebra become small non-zero constants. More complicated algebras can be created by incoroporating operator dependence in the commutation algebra, such as is predicted by Snyder space and by the minimal length uncertainty relations.

Since the field of noncommutative quantum mechanics studies several deformations of the standard algebra, we adopt a nomenclature for referring to the most common algebras that are considered in the literature. Algebras that result from deforming Eq.(1.3) into Eq. (5.1) while leaving Eq. (1.2) and Eq. (1.4) unaltered are referred to as "Space-Space" Algebras (SSA) since they only affect the spatial components of the commutation algebra. Similarly, algebras that only alter Eq. (1.4) to become Eq. (5.2) are referred to as "Momentum-Momentum" Algebras (MMA). Finally, algebras that adopt both deformations are called "Space-Momentum" Algebras (SMA).

Certainly Snyder space does not fall into any of the three categories mentioned above. In fact, with a little imagination, one may consider many other possible deformations of the Heisenberg algebra. In this chapter we consider two other commutation algebras that do not fall into any of the classes mentioned above. One is known as the Minimal Length Uncertainty Relations (MLUR) and the other as Dynamical Quantization (DQ). We present these particular algebras because of their connection to the commutation algebra of Snyder space.

Many of these algebras may seem at first unrelated to Snyder space and, therefore, out of place in this work. However, we were originally motivated to study Snyder space because of the intriguing results that appeared in the noncommutative quantum mechanics literature. We set out to see if similar results could be produced in Snyder space. The discussion in this chapter places the results of this thesis within their wider framework of noncommutative quantum mechanics and establishes a bridge to much of the current literature. In addition, we discuss the relation between Snyder space and classical mechanics in section 5.6.

5.2 Space-space noncommutative quantum mechanics

5.2.1 Motivation

As mentioned previously, the field of noncommutative quantum mechanics was born with the low-energy limit prediction of string theory that modifies the usual commutation algebra among position operators of different dimension, x and y, to Eq. (5.1). The SSA is by far the most common algebra studied in noncommutative quantum mechanics, as one can see by briefly scanning the literature on the subject.

5.2.2 Results

From the many interesting results that come from studying the SSA, I choose to discuss one that pertains to the two-dimensional SHO. Other results can be found in [21].

In noncommutative quantum mechanics the two-dimensional SHO, is sometimes referred to as the SHO in the noncommutative plane. Unlike Snyder space, the

noncommutative plane represents the simplest interesting case. Indeed, there are no interesting one-dimensional problems in SSA, since the one-dimensional algebra is identical to that of standard quantum mechanics. However, as we see below, the SHO in the noncommutative plane is equivalent to the SHO superimposed on a constant magnetic field in standard quantum mechanics [17].

Following the techniques of operator transformation similar to those discussed in chapters 3 and 4 of this work, we can transform the two-dimensional SHO from the noncommutative plane into a perturbed problem in standard quantum mechanics. In particular, we introduce operators $\tilde{x}$, $\tilde{y}$, $\tilde{p}_x$, and $\tilde{p}_y$ satisfying

$$[\tilde{x}, \tilde{p}_x] = i\hbar \tag{5.3}$$

$$[\tilde{y}, \tilde{p}_y] = i\hbar, \tag{5.4}$$

with all other commutators vanishing. It follows from this definition that the operators given by

$$x = \tilde{x} - \tilde{p}_y \frac{\Theta}{2\hbar} \tag{5.5}$$

$$y = \tilde{y} + \tilde{p}_x \frac{\Theta}{2\hbar} \tag{5.6}$$

$$p_x = \tilde{p}_x \tag{5.7}$$

$$p_y = \tilde{p}_y, \tag{5.8}$$

satisfy

$$[x, p_x] = [y, p_y] = i\hbar \tag{5.9}$$

$$[x, y] = i\Theta \tag{5.10}$$

$$[p_x, p_y] = 0. \tag{5.11}$$

$$\tag{5.12}$$

Using some straightforward algebra, the SHO Hamiltonian H can be transformed

from its usual expression

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m\omega^2\left(x^2 + y^2\right), \tag{5.13}$$

to a new form

$$H = \frac{1}{1+m^2\omega^2\Theta^2}\left(\frac{\left(\tilde{p}_x - \frac{e}{c}A_x\right)^2 + \left(\tilde{p}_y - \frac{e}{c}A_y\right)^2}{2m} + \frac{m\omega^2}{2\left(1+m^2\omega^2\Theta^2\right)}\left(\tilde{x}^2 + \tilde{y}^2\right)\right), \tag{5.14}$$

which includes an effective vector potential

$$A_x = \frac{m^2\omega^2\Theta c}{e\left(1+m^2\omega^2\Theta^2\right)}\tilde{y} \tag{5.15}$$

$$A_y = -\frac{m^2\omega^2\Theta c}{e\left(1+m^2\omega^2\Theta^2\right)}\tilde{x} \tag{5.16}$$

and a modified mass and frequency.

One can calculate the magnetic field that results from this effective potential. It is a constant field perpendicular to the plane. Thus, the two dimensional SHO in the SSA is equivalent to a simple harmonic oscillator in standard quantum mechanics superimposed on a constant magnetic field. From a modification of the dynamics we obtain a new interaction. This can be considered an explanation of magnetism. Alternatively, one can interpret the effect of magnetism in quantum mechanics to introduce non-commutativity among spatial coordinate operators. This result is one of the principle results from noncommutative quantum mechanics and was one of our principle motivations for studying the SHO in Snyder space.

5.2.3 Relation to Snyder space

It may appear that there is little relation between Snyder space and the SSA. A quick glance at the two commutation algebras reveals that Snyder space is much more complicated. Furthermore, it is clear that the effective Hamiltonians resulting

from the transformations in Eqs. (5.5) - (5.8) will be radically different from those resulting from Eqs. (4.17) - (4.20). We see this explicitly in the case of the two-dimensional SHO. The result of the previous section results from the fact that the transformation of Eqs. (5.5) - (5.8) is linear. Thus, the SHO in the SSA does not introduce higher order terms in either position or momentum when transformed into standard quantum mechanics, while the mixing of position and momentum terms translates into a magnetic field. In the case of Snyder space, the transformation in Eqs. (4.17) - (4.20) introduce higher order terms in momentum and higher-order mixed terms. As a consequence the result from the SSA that equates noncommutativity with a constant magnetic field cannot be reproduced in Snyder space.

The SSA introduces constant non-zero commutation relations among coordinate position operators; therefore, the SSA is much simpler than the Snyder algebra because it does not consider operator dependence. Alternatively, since Θ_{ij} is a real antisymmetric matrix, in D dimensions, the SSA has $D(D-1)/2$ independent parameters to be determined. By contrast, Snyder space has only one parameter to be fixed, a. In this sense, Snyder space is simpler than the SSA. This is not a simplification in complexity, but in experimental verifiability.

Still, we point out two important similarities between Snyder space and the SSA. First, the momentum operators corresponding to different dimensions commute in both cases. Thus, the form of the kinetic energy term in both spaces is essentially identical to the kinetic energy term in standard quantum mechanics. Secondly, and more importantly, both have been introduced to extend the description of real interactions in a way to avoid perceived problems in the current model. Snyder's original formulation of Snyder space was to solve a particular problem in quantum field theory. The SSA arises as the low-energy limit of string theory, and could prove to be its first experimental tests. In fact, Snyder's original work is often cited in papers on

noncommutative quantum mechanics because of its historical precedence in being the first serious formulation of quantum mechanics on a noncommutative manifold, even though the exact form of Snyder space has little relevance to the SSA.

5.3 Alternate forms of noncommutative quantum mechanics

5.3.1 Momentum-momentum and space-momentum noncommutative quantum mechanics

The MMA refers to commutation algebra described by Eq. (5.2). As far as we have been able to tell, this commutation relation has been postulated as an analog to the SSA. As such, it has been studied as an exercise in mathematical physics and not in order to solve a particular shortcoming of the current theory. This may explain why there is less work done in this field.

The SMA, refers to the combination of the SSA and the MMA. As in the case of the MMA, the SMA is a generalization of the SSA. It is the full generalization of the Heisenberg algebra that does not introduce operator dependence in the commutators since in the SMA algebra all vanishing commutators have been modified to some non-zero constant. This version of noncommutative quantum mechanics is studied more than the MMA, but it still does not draw the attention that SSA does. This is likely a consequence of the SSA being a prediction of string theory while the other two are merely the logical generalization of the SSA.

5.3.2 Results

In the case of the MMA, it is straightforward to show that the transformation

$$x_i = \tilde{x}_i \tag{5.17}$$

$$p_i = \tilde{p}_i - \tilde{x}_l \frac{\tilde{\Theta}_{il}}{2\hbar} \tag{5.18}$$

satisfies Eq. (5.2), where $\tilde{x}_i$ and $\tilde{p}_i$ satisfy the Heisenberg algebra and the Einstein summation convention has been used.

Given the transformation above, it follows that for any Hamiltonian, it is equivalent to a Hamiltonian in standard quantum mechanics with the same potential energy function, but with a constant magnetic field. In particular,

$$H = \frac{p_i p_i}{2m} + V(x_i) = \frac{(\tilde{p}_i - \frac{e}{c}A_i)(\tilde{p}_i - \frac{e}{c}A_i)}{2m} + V(x_i), \tag{5.19}$$

where

$$A_i = \tilde{x}_l \frac{\tilde{\Theta}_{il} c}{2e\hbar}. \tag{5.20}$$

Thus the result from the SSA that equates the SHO to an SHO with a constant magnetic field is fully general in the MMA. It is generalized both in the sense that it includes all potential energy functions and that it is valid in three dimensions.

Since the SMA is the combination of the SSA and MMA, it follows that the two-dimensional SHO in the SMA is also equivalent to the SHO superimposed on a constant magnetic field perpendicular to the plane.

In standard quantum mechanics there is a symmetry between position and momentum. In these deformations of the Heisenberg algebra, this symmetry is broken. In standard quantum mechanics this symmetry is also broken through minimal coupling, in which momentum is transformed by

$$p \to p - \frac{e}{c}A. \tag{5.21}$$

There is no similar "minimal coupling" transformation for position. The broken symmetry in the commutation algebra is equivalent to the broken symmetry caused by minimal coupling.

5.3.3 Relation to Snyder space

If the SSA is related to Snyder space through a matter of principle, i.e. that they both have been proposed as viable physical theories, then it follows that there is little relation between Snyder space and the MMA and SSA. The SMA is more complicated than the SSA; however, it is not nearly as complicated as the Snyder-space algebra in the sense that it does include operator dependence in the commutation relations.

When considering the number of parameters, both the MMA and the SMA algebras are more complex than the Snyder algebra. In D dimensions, the MMA has $D(D-1)/2$ independent parameters and the SMA has $D(D-1)$ independent parameters. As mentioned in section 5.2.3, this is a simplification in the verifiability and not in computational complexity.

5.4 Minimal length uncertainty relations

Another algebra that we consider in this chapter is the minimal length uncertainty relations. The MLUR were proposed by Kempf [10], although it appears that it was also a prediction of string prior to Kempf's derivation. The commutation relations are derived by considering the uncertainty relation given by

$$\frac{\Delta x}{x_0} \geq \frac{1}{2}\left(\frac{p_0}{\Delta p} + \frac{\Delta p}{p_0}\right), \tag{5.22}$$

from which it follows that Δx has a minimum value of x_0 at $\Delta p = p_0$, as can be seen in Fig. 5.2.

Figure 5.2 The minimal length uncertainty relations result in a minimal uncertainty in length at $x = x_0$

Uncertainty relations between operators are related to the commutator by the expression

$$\Delta A \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|. \tag{5.23}$$

Furthermore, if we make the approximation,

$$\langle p \rangle \approx 0, \tag{5.24}$$

then

$$\Delta p^2 = \langle p^2 \rangle - \langle p \rangle^2 \approx \langle p^2 \rangle, \tag{5.25}$$

and it follows that the commutation relation

$$\left[\frac{x}{x_0}, \frac{p}{p_0} \right] = i \left(1 + \frac{p^2}{p_0^2} \right) \tag{5.26}$$

leads to Eq. (5.22). By requiring that $[x, p] \approx i\hbar$ in the low-energy limit, it follows that $x_0 p_0 = \hbar$. Finally, since x_0 is the minimal length that can be resolved, we can set $x_0 = a$, i.e. the fundamental length of Snyder space, and we have reproduced one-dimensional Snyder space.

Kempf generalizes the minimal length uncertainty relations to multiple dimensions as follows

$$[x_i, p_j] = i\hbar \left(\left(1 + \gamma p^2 \right) \delta_{ij} + \gamma' p_i p_j \right) \tag{5.27}$$

$$[p_i, p_j] = 0. \tag{5.28}$$

The commutator between position operators is uniquely determined by the Jacobi identity, and is given by

$$[x_i, x_j] = i\hbar \frac{(2\gamma - \gamma') + (2\gamma + \gamma')\,\gamma p^2}{1 + \gamma p^2} \left(p_i x_j - p_j x_i \right) \tag{5.29}$$

The minimal length uncertainty relations are also related to string theory, where it has been suggested there is a minimal length below which resolution is impossible [10]. The commutation relations given above describe this behavior. This is yet another motivation for considering this algebra.

5.4.1 Results

The principal result from the MLUR involves the simple harmonic oscillator. Kempf first calculated the first order energy corrections to the SHO after postulating the commutation relations [11]. Chang *et al.* later found an exact solution in arbitrary dimensions [12]. The result is given below for D dimensions and angular momentum, L:

$$\begin{aligned} E_n = \hbar\omega \left[\left(n + \frac{D}{2} \right) \sqrt{1 + \left(\gamma^2 L^2 + \frac{(D\gamma + \gamma')^2}{4} \right) m^2 \hbar^2 \omega^2} \right] \\ + \hbar\omega \left[\left((\gamma + \gamma') \left(n + \frac{D}{2} \right)^2 + (\gamma - \gamma') \left(L^2 + \frac{D^2}{4} \right) + \gamma' \frac{D}{2} \right) \frac{m\omega\hbar}{2} \right]. \end{aligned} \tag{5.30}$$

The relation between this formula and the energy formulas that I present in chapters 3 and 4 are discussed in the next section.

5.4.2 Relation to Snyder space

The algebra derived from the minimal length uncertainty relations is the closest related to Snyder space of all algebras considered in this chapter. In fact, the MLUR is a generalization of the Snyder space commutation algebra. Kempf derived the algebra independent of Snyder's work and from an entirely different premise than that of Snyder. His method was analogous to the one presented in section 5.4.1. The fact the MLUR was derived independently of Snyder is another motivation for reconsidering Snyder space.

The exact relation between Snyder space and the MLUR can be seen by considering Eqs. (5.27) - (5.29) and making the identification $\gamma' = \alpha$ and $\gamma = 0$, where $\alpha = a^2/\hbar^2$ as we defined in chapter 2. In this case we recover the Snyder space commutation algebra. Thus all the results of the minimal length uncertainty relations can be applied to Snyder space through this identification. The results of Snyder space are not generally applicable to the MLUR since the MLUR is more general than the Snyder algebra.

If we consider Eq. (5.30) and let $\gamma' = \alpha$ and $\gamma = 0$, we find that the energy of a a D dimensional isotropic SHO in Snyder space is given by:

$$E_n = \hbar\omega \left[\left(n + \frac{D}{2} \right) \sqrt{1 + \left(\frac{\beta}{2} \right)^2} \right] + \frac{\hbar\omega}{2} \left[\left(\beta \left(n + \frac{D}{2} \right)^2 + \beta \left(L^2 + \frac{D^2}{4} \right) + \beta \frac{D}{2} \right) \right] \tag{5.31}$$

The MLUR are the only commutation relations considered in this chapter that are more complicated than the Snyder algebra both in the appearance of operators in the commutation relations and in the number of parameters that define the algebra, although the complexity is comparable in both cases. The MLUR differ from the Snyder commutation algebra by the appearance of terms proportional to the square

of the total momentum p^2 which are always accompanied by a factor of γ. It also is characterized by the appearance of two parameters in the commutation relations γ and γ'. We have seen that γ' plays the role of α in Snyder space and that the MLUR reduce to the Snyder algebra in the case that $\gamma = 0$.

5.4.3 Applications

When Kempf originally postulated the commutation algebra in Eqs. (5.27) - (5.29), he conjectured that it may prove useful in studies of quantum gravity where it had been suggested that spacetime is quantized. However, since the experimental verification of quantized spacetime is so remote, he also suggested that the commutation relations could be useful in describing non-pointlike particles, such as nucleons or strings. Kempf suggests that applying this deformed algebra to the hydrogen atom could yeild an upper bound for the nonpointlike behavior of the electron.

Chang *et al.* also considered applying the SHO energy spectrum to an electron in a Penning trap to probe the upper limit on the minimal length. Restrictions of the applicability of the energy spectrum arise because of the nonrelativistic kinetic energy assumption. They estimate that a minimal length on the order of 10^{-16} m could be resolved using this method [12].

I am not aware of any research that uses the MLUR to study non-pointlike particles. Nor am I aware of any experiments that have tried to resolve a minimal length using the modified energy spectrum described above.

5.5 Dynamical quantization

The final algebra that we consider in this chapter is known as Dynamical Quantization. It is obtained by modifying the Heisenberg algebra to become

$$[x_i, p_j] = \left(i\hbar + i\frac{l}{c}H \right) \delta_{ij}, \tag{5.32}$$

where H is the Hamiltonian of the system being considered, l is some fundamental unit of length, and c is the speed of light. All other commutation relations are unmodified. From this definition it should be clear why this algebra is known as dynamical quantization: the dynamics of the system define the commutation relations.

DQ was first proposed by Saavedra *et al.* in the early 1980s as a method of describing high-energy interactions such as quark interactions [15, 16]. Little work was done on the subject, however, and nothing else was published on DQ until this year (2006), when renewed interest was generated by the study of other modified commutation relations [22].

5.5.1 Results

In two brief letters in 1981 and 1985, Saavedra *et al.* report a number of results obtained using DQ [15, 16]. These results include the simple harmonic oscillator and infinite square well spectra. The SHO spectrum is obtained in the relativistic case, so it is not directly comparable to other results reported in this work. The infinite square well spectrum, which is nonrelativistic, is interesting in that it only had a finite number of energy levels, which is very different from the standard case.

The energy spectrum of the infinite square well should be related to the infinite square well in Snyder space. In DQ, when the system is a nonrelativistic free particle, the commutation algebra is essentially that of Snyder space, as discussed in the next subsection. Since the infinite square well is essentially a free particle confined to a

finite region, we expect a similar result to be found in Snyder space. In chapter 2, we mentioned that the infinite square well could not be formulated in terms of differential equations. However, the methods of operator transformations can be used to solve the infinite square well in Snyder space. The method is described in section 5.5.3, where the energy spectrum is also given.

5.5.2 Relation to Snyder space

As Saavedra *et al.* observe, in the case of the one-dimensional, nonrelativistic free particle, the commutation relation of DQ is essentially the same as that of Snyder space [16]. This claim is correct in the sense that both commutation relations add a term proportional to the square of the momentum; however, there are some fundamental differences that should be pointed out. First, Snyder's commutation relation is the same for all particles and the additional term in the commutation relation is proportional to the square of the fundamental length. In contrast, for the free particle in DQ the additional term is proportional to the fundamental length and inversely proportional to the mass of the particle. That is, the operator dependence of the two commutation relations is identical, but the way the constants of nature enter the commutation relations is very different.

Regarding their relative complexity, Snyder space and DQ are comparably complex. They both involve operator dependence in the commutation relations, although the operator dependence in DQ varies from system to system. Both algebras introduce only one new parameter into the formulation and in both cases it is a fundamental unit of length. In this respect, DQ and Snyder space are very similar.

5.5.3 One-dimensional infinite square well

For a particle in a one-dimensional infinite square well in Snyder space, the Hamiltonian is

$$H = \frac{p_S^2}{2m} + V(x_S), \tag{5.33}$$

where $V(x_S) = 0$ for $0 \leq x_S \leq L$ and infinite elsewhere. This amounts to finding the eigenvalues of the operator

$$H = \frac{p_S^2}{2m} \tag{5.34}$$

with wave functions that vanish at the boundaries of the well.

In order to apply the techniques of operators transformations developed in chapter 3, we cannot apply any of the transformations in Eq. (3.6). If we did apply such a transformation, our next step would be create a perturbation series in α; however, since the potential energy function is not analytic, no such series exists for these transformations. Therefore, we choose the transformation in Eq. (3.7) as a tool for analyzing this system. In this case, the Hamiltonian becomes

$$H = \frac{\tan^2(\sqrt{\alpha}p)}{2m\alpha} = \frac{p^2}{2m} + \alpha\frac{p^4}{3m} + ... \tag{5.35}$$

Using this transformation, only even powers of p appear. Therefore, this Hamiltonian has common eigenstates with the Hamiltonian for the infinite square well in standard quantum mechanics, $H_0 = p^2/2m$, with the same boundary conditions. Hence, we can sum the the perturbation series exactly by noting that

$$H = \frac{\tan^2\left(\sqrt{2m\alpha H_0}\right)}{2m\alpha}. \tag{5.36}$$

Therefore, the energy eigenvalue spectrum is given by

$$E_n = \frac{\tan^2\left(\sqrt{2m\alpha E_{0n}}\right)}{2m\alpha}, \tag{5.37}$$

where E_{0n} are the eigenvalues of the unperturbed infinite square well in standard quantum mechanics. It is well-known that $E_{0n} = \frac{n^2\hbar^2\pi^2}{2mL^2}$. For convenience, we introduce the dimensionless parameter λ

$$\lambda = \frac{L}{a} \tag{5.38}$$

where a is the fundamental length in Snyder space. Recalling that $\alpha = \frac{a^2}{\hbar^2}$, we find that the energy spectrum in Snyder space is

$$E_n = \frac{\tan^2(n\pi/\lambda)}{2m\alpha}. \tag{5.39}$$

If the transformation of Eq. (3.7) is to be invertible, we must truncate the range of the operator p. This requirement is not pertinent to the methods of chapter 3, but in this analysis it becomes very important. Therefore, we have

$$-\frac{\pi}{2} \leq \sqrt{\alpha}p \leq \frac{\pi}{2}, \tag{5.40}$$

and the quantum number n in Eq. (5.39) does not range over all positive integers in the Snyder space spectrum. Instead, it has the range $n = 1, 2, ..., n_{max}$ where

$$n_{max} = \lfloor \lambda/2 \rfloor, \tag{5.41}$$

where $\lfloor x \rfloor$ denotes the largest integer less than x. Thus, the infinite square well has only a finite number of energy levels, which is very different from the standard case. The spectrum derived in this section is the same as that presented by Saavedra *et al.* in [15].

5.6 The classical limit

To conclude this discussion regarding the relation Snyder space holds with other modified theories of nature, we discuss the relation between Snyder space and classical mechanics, or rather, how the classical limit becomes modified in Snyder space.

Roughly speaking, the classical limit in standard quantum mechanics is the limit where $\hbar$ approaches zero, in which limit, the results of classical mechanics are recovered from their quantum analogs. Thus the classical limit is essentially the limit where the noncommutative operators of quantum mechanics become commuting dynamical variables and the commutator is replaced by the Poisson bracket.

In the case of Snyder space we expect a similar limit to exist, but it must be augmented by the limit that a approaches zero in order to recover the continuum of classical mechanics. We refer to this last limit as the continuum limit and the limit that $\hbar$ approaches zero as the classical limit. It should be clear from the commutation algebra of Snyder space that standard quantum mechanics is recovered in the continuum limit.

By inspecting the commutation relations of Snyder space, we see that the factor $\hbar\alpha = a^2/\hbar$ appears frequently. As a consequence to recover classical mechanics from Snyder space it is necessary to take the continuum limit before taking the classical limit. If one takes the classical limit while holding a constant, then the commutation relations of Snyder space become singular. That is, the classical limit only exists after taking the continuum limit.

Because of the existence of the terms proportional to $a^2/\hbar$ there exists another limit that recovers neither standard quantum mechanics nor classical mechanics. If a and $\hbar$ vanish simultaneously such that $a^2/\hbar$ remains constant, the result would be a commutation algebra that is neither standard quantum mechanics nor classical mechanics. The consequences of this limit have not been investigated, but the results are likely to be interesting. It could be that this limit represents a formulation of classical mechanics in quantized spacetime or a type of noncommutative classical mechanics, although this guess is purely speculative.

Chapter 6

Conclusions

6.1 Summary of results

In this thesis we have investigated one-dimensional and two-dimensional Snyder space and have performed a detailed analysis of the simple harmonic oscillator in Snyder space. The major result of this thesis is the development of a methodology and mathematical tools to analyze physical systems in Snyder space.

In chapter two, we studied the details of one and two-dimensional Snyder space and developed a mathematical description for quantum mechanical operators in both the underlying space and the momentum space representations. In one dimension, we showed that in the underlying space, the Fourier series decomposition of the wave function results in a discrete position space representation. In two dimensions, we showed that the radial position operator, $r = \sqrt{x^2 + y^2}$, is related to the spherical harmonics and has eigenvalues $\sqrt{n}a$, for all non-negative integers n. In addition, we showed that both the underlying space and the momentum space representation of the energy eigenfunction for the free particle are delta functions and that the form of the solution in momentum space is identical to that of standard quantum mechanics.

In chapters 3 and 4, we presented a method for transforming a system in Snyder space to the analogous system in standard quantum mechanics. The appearance of additional interactions terms is consistent with the generation of magnetic fields in noncommutative quantum mechanics . We also demonstrated how to formulate exact energy eigenvalue problems in Snyder space in terms of differential equations. The existence of the underlying space results in two differential equations existing for each system, which has the advantage of presenting two methods for obtaining solutions for each system.

The methods described above were applied to the SHO in one and two dimensions in chapters 3 and 4 respectively. We worked out the perturbation on the spectrum to fourth order in one dimension and second order in two dimensions. The validity of the methodology is vindicated by the success in finding the energy spectrum and the energy eigenstates for both systems.

Finally, we described the relation Snyder space has with other areas of noncommutative quantum mechanics. While there is little similarity between the forms of the commutation algebras, Snyder space is certainly related to these algebras for its historical precedence. There are other modern algebras, such as the minimal length uncertainty relations and dynamical quantization that are much more closely related to Snyder space, suggesting that Snyder space is still relevant today.

6.2 Outlook

Since Snyder space is an alternate formulation of quantum mechanics, all of the problems usually considered in quantum mechanics are possible avenues of future research. In the final section of chapter 2, I discuss many of the possible potentials that could be considered. Some of the potentials mentioned included other polynomial

potentials and inverse power potentials such as the Coulomb potential. In principle any potential that can be expressed as an analytic function of the position could be considered. Potentials that are defined as piecewise functions of the position have an added complication that they result in differential equations that are defined piecewise on a differential operator. Such systems are inherently more complicated in Snyder space, although they can be solved as we have demonstrated by finding the energy spectrum of the infinite square well in section 5.5.3.

I have only considered one and two-dimensional problems in this work. In chapter 4, I commented briefly on the increasing complexity with each additional dimension. Since the physical world is really four-dimensional, additional work could be done to study Snyder space with an additional spatial dimension. Snyder space also inherently incorporates a time operator. The role of time and Dynamics in Snyder space should prove to be an interesting research subject.

In this work, we studied the simple harmonic oscillator; a natural extension of this work could be to study other types of oscillators. Specific examples could include coupled oscillators, damped oscillators, and driven oscillators.

Having restricted our attention to bound states in this thesis, a different avenue of research could be to study unbound states and scattering potentials in Snyder space. Since discrepancies between Snyder space and the standard theory are likely to manifest themselves at high energies, high-energy scattering experiments could be used as an experimental verification of Snyder space.

In addition to studying specific systems of Snyder space, there is still work to be done understanding the relation between Snyder space and other theories of noncommutative quantum mechanics . In particular, the minimal length uncertainty relations are obviously very closely related to Snyder space. While it has been claimed that this commutator algebra is related to string theory [10], I have not found the precise

relationship. If Snyder space can be predicted by string theory, it would be a strong motivation to reconsider Snyder space.

Clearly, there is much work that could be done in studying Snyder space. This work has only studied the most fundamental aspects of the space, but many of the consequences have yet to be worked out. Given the enormous amount of literature on the subject of deformed algebras, it is certainly appropriate that the study of Snyder space continue in hopes of helping to resolve some of the dissatisfactions with the current theory.

Bibliography

[1] H. S. Snyder, "Quantized Space-Time," Phys. Rev. **71,** 38–41 (1947).

[2] H. S. Snyder, "The Electromagnetic Field in Quantized Space-Time," Phys. Rev. **72,** 68–71 (1947).

[3] B. Carazza and H. Kragh, "Heisenberg's lattice world: the 1930 theory sketch," Am. J. Phys **63,** 595–605 (1995).

[4] E. L. Hill, "Relativistic Theory of Discrete Momentum Space and Discrete Space-Time," Phys. Rev. **100,** 1780–1783 (1955).

[5] L. Smolin, *Three Roads to Quantum Gravity* (Basic Books, New York, NY, 2005).

[6] D. Meschini, "Planck-scale physics: facts and beliefs," gr-qc/0601097 (2006).

[7] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd ed. (Kluwer Academic Publishers, Boston, MA, 1999).

[8] A. C. de la Torre, "The ubiquitous XP Commutator," Eur. J. Phys. **27,** 225–230 (2005).

[9] S. Dulat and K. Li, "Commutator Anomaly in Noncommutative Quantum Mechanics," hep-th/0508060 (2005).

[10] A. Kempf, G. Mangano, and R. B. Mann, "Hilbert Space Representation of the Minimal Length Uncertainty Relation," Phys. Rev. D **52,** 1108 (1995).

[11] A. Kempf, "Nonpointlike Particles in Harmonic Oscillators," hep-th/9604045 (1996).

[12] L. N. Chang, D. Minic, N. Okamura, and T. Takeuchi, "Exact solution of the harmonic oscillator in arbitrary dimensions with minimal length uncertainty relations," Phys. Rev. D **65,** 125027 (2002).

[13] I. Dadic, L. Jonke, and S. Meljanac, "Harmonic Oscillator with minimal length uncertainty relations and ladder operators," Phys. Rev. D **67,** 087701 (2003).

[14] G. Jaroszkiewica, "A dynamical model for the origin of Snyder's quantized space-time algebra," J. Phys. A:Math Gen **28,** L343–L347 (1995).

[15] S. Montecinos, I. Saavedra, and O. Kunstmann, "High Energy Quantum Mechanics and Dynamical Quantization," Phys. Lett. A **109,** 139 (1985).

[16] I. Saavedra and C. Utreras, "A generalization of Quantum Mechanics for High Energies and Quark Physics," Phys. Lett. B **98,** 74 (1981).

[17] H. O. Girotti, "Noncommutative quantum mechanics," Am. J. Phys. **72,** 608–612 (2004).

[18] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists*, fifth ed. (Harcourt Academic Press, San Diego, CA, 2001).

[19] J. S. Townsend, *A Modern Approachh to Quantum Mechanics* (University Science Books, Sausalito, CA, 2000).

[20] F. W. Olver, *Asymptotics and Special Functions* (Academic Press, New York, 1974).

[21] L. Castellani, "Noncommutative geometry and physics: a review of selected recent results," hep-th/0005210 (2000).

[22] H. Calisto and C. A. Utreras-Diaz, "On Dynamical Quantization," quant-ph/0603246 (2006).

[23] K. S. Thorne, *Black Holes and Time Warps: Einstein's Outrageous Legacy* (W. W. Norton and Company, New York, 1994).

[24] J. R. Oppenheimer and H. S. Snyder, "On Continued Gravitational Contraction," Phys. Rev. **56,** 455–459 (1939).

[25] M. K. Transtrum and J. F. S. Van Huele, "Commutation relations for functions of operators," J. Math. Phys. **46,** 063510 (2005).

[26] D. J. Griffiths, *Introduction to Quantum Mechanics* (Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1995).

[27] R. W. Robinett, *Quantum Mechanics: Classical Results, Modern Systems, and Visualized Examples* (Oxford University Press, New York, 1997).

[28] N. Wheeler, Reed College. Private Communication (2005).

Appendix A

Biographical Information on Hartland S. Snyder

In this appendix we collect the relatively few facts that we were able to find on Hartland S. Snyder. Considering the importance of at least three of his contributions in very different fields, this lack of available information is surprising and unfortunate.

Hartland S. Snyder was a native of Utah, being born in 1913 in Salt Lake City. It has been rumored that before studying physics he had worked as a truck driver [23]; however, he graduated from the University of Utah in 1937, at the age of 24, so he certainly did not find physics late in life.

Snyder studied at the University of California at Berkeley under J. Robert Oppenheimer. His graduate work included a paper that was pivotal in the theoretical discovery of black holes [24].

Shortly following his graduate studies, Snyder accepted a faculty position at Northwestern University in 1940, where he remained until 1947. It was during his final period at Northwestern that he published his work on quantized spacetime, which is today known as Snyder space. Snyder's work on the subject appears in two papers,

and it is not known why he did not pursue the subject further. It is possible that the project was abandoned because later that year he left Northwestern to work at Brookhaven National Laboratory.

Snyder spent the rest of his life working at Brookhaven. His work at the national laboratory was centered on accelerator design. Together with Courant and Livingston, Snyder was instrumental in discovering the "strong focussing" principle which is used in nearly all modern accelerators.

Apart from being an influential physicist, Snyder had a colorful character. Kip Thorne reports that he clashed with other members of Oppenheimer's research group while at Berkeley, who came from middle and upper class families and had much more refined tastes that Snyder. Later in his life, Snyder bet Maurice Goldhaber $500 that the antiproton exists. Snyder of course won the bet when the anti-proton was discovered experimentally, although he never cashed the check.

Snyder died at a relatively young age in 1962 after having suffered a heart attack while on sabbatical at California's Lawrence Radiation Laboratory.

Appendix B

Algorithms for Manipulation of Noncommutative Objects

This appendix contains technical material about the implementation of algorithms in Maple that were instrumental in obtaining the results described in this thesis. This material is presented for completeness, to facilitate reproducing my results and continuing Snyder space studies.

B.1 Commutator algorithms

As mentioned in chapters 3 and 4, the expressions that result from the operator transformations can be quite complex, and for this purpose I developed algorithms to manipulate such expressions. The algorithms are based upon an algorithm for evaluating the commutator. When the commutation algebra is made up of operators that have constant commutation relations, then the algorithm implements the formula that is derived in [25] and is restated below.

Given n operators $x_1, ..., x_n$ such that $[x_i, x_j] = c_{ij}$, where c_{ij} are c-numbers, then

$$[f(x_1, ..., x_n), g(x_1, ..., x_n)] = \underbrace{\sum\sum\sum ... \sum}_{k_{1,2}\ k_{1,3}\ k_{2,3}\ \ k_{n-1,n}} \left(\prod_{j=2}^{n} \prod_{i=1}^{j-1} \frac{(-c_{ij})^{k_{ij}}}{k_{ij}!} \right) \times \left(\partial_{x_1}^{k_1} ... \partial_{x_n}^{k_n} g \partial_{x_1}^{k'_1} ... \partial_{x_n}^{k'_n} f - \partial_{x_1}^{k_1} ... \partial_{x_n}^{k_n} f \partial_{x_1}^{k'_1} ... \partial_{x_n}^{k'_n} g \right), \tag{B.1}$$

where

$$k_i = \sum_{j=i+1}^{n} k_{ij} \tag{B.2}$$

and

$$k'_i = \sum_{j=1}^{i-1} k_{ji}. \tag{B.3}$$

The index of each summation ranges from zero to infinity ($i = 0...\infty$) with the restriction that the indices within the underbrace ($\underbrace{\ }$)are not all simultaneously zero.

When the commutation algebra is not made up of operators with constant commutation relations, then the algorithm expands the commutator into fundamental commutators using properties such as linearity

$$[\alpha A + \beta B, C] = \alpha[A, C] + \beta[B, C], \tag{B.4}$$

Leibnitz' rule

$$[AB, C] = A[B, C] + [A, C]B, \tag{B.5}$$

and antisymmetry

$$[A, B] = -[B, A], \tag{B.6}$$

where we have denoted operators by capital letters, A, B, and C and constants by Greek letters α and β.

Other algorithms make use of the commutator algorithm to perform basic tasks such as ordering, which is rewriting an expression such that all occurrences of a

specific operator appear to the left, following by all occurrences of another operator, etc.

In addition, there are several algorithms that were developed to generalize functions that already exist in Maple for the star, $*$, or commutative product to include the non-commutative product.

The commutator algorithm is implemented as a Maple module. As such, several algorithms are written as internal procedures to which the user does not have access, while exported procedures can be used by the user. There are also several global variables that the user must define in order to specify the form of the commutation algebra. What follows is a list of all the user-defined global variables, user available algorithms, the syntax for calling the algorithms, and a description of the the purpose of each.

B.1.1 Global variables

```
X
```

```
X is a list.  The name of each element is considered an operator
by the algorithms in the module.  The order of the elements of X
defines the "normal order" for the algorithms normalorder(arg) and
Normalorder(arg).
```

```
C
```

```
C is an antisymmetric matrix that defines the commutation algebra.
In particular com(X[i],X[j]) = C[i,j].
```

B.1.2 Algorithms, syntax, and descriptions

```
com(arg1, arg2)

arg1,arg2 - expressions

Evaluates the commutator of arg1 and arg2 from fundamental
commutation relations in one of two ways.  If arg1 and arg2 are
constructed from operators each of which commute with their
commutators, then com uses a sophisticated formula involving
partial derivatives to evaluate the commutator.  Otherwise, com
uses the properties of linearity and Leibnitz' rule to express the
commutator as a sum and product of fundamental commutators, which
can be evaluated from the fundamental commutation relations.


convert(arg, `.`)

arg - expression

If type(arg,`^`); returns true, (that is, if arg is an expression
involving `^`) then arg is converted to a dot product assuming
that the carrot operator implied non-commutative multiplication.
Otherwise, arg is converted a list, which is subsequently
converted to a non-commutative product.

deg(arg,op)
```

```
arg - expression

op - variable name

If arg is a polynomial of non-commutative products in the variable
op, then deg(arg,op); returns degree of the polynomial.
Otherwise, deg(arg,op) returns degree(arg,op);

Depend(arg)

arg - expression

Depend(arg) returns a list of all elements X that appear in arg.

distribute(arg)

arg - expression

Distribute(arg) distributes all non-commutative products over
addition.

dotProd(arg, k = m..n)

arg - expression
```

```
k - name - the product index

m,n - integers or arbitrary expressions

dotProd(f,k=m..n) returns the expression f evaluated at k = m
multiplied non-commutatively by f evaluated at k = m + 1 and so on
through f evaluated at k = n.  It is analogous to the procedure
product(f, k = m..n) defined by Maple, with the exception that it
employs a non-commutative product in place of the commutative
case.

expand(arg)

arg - expression

Every instance of the expression Com(a,b) in args is expanded
using the properties of the commutator, linearity and Leibnitz'
rule specifically.

LoadAlg(int)

int - integer value 1,2,3, 4

LoagAlg(n) defines X and C to be the n-dimensional Heisenberg
algebra, n = 1,2,3.  In addition, LoadAlg defines a global
variable _ALG that the com procedure recognizes and employs a
```

```
faster algorithm for evaluating the commutator.  LoadAlg(4)
defines X and C to take values for the angular momentum
commutation algebra in three dimensions.

Normalorder(arg)

arg - expression

Normalorder(arg) uses the relation a.b = b.a - Com(b,a) to change
the order of operators in the expression, arg.  The preferred
order is determined by the user defined list X.  Every instance of
the first element of X, X[1], is commuted to the left, and so
forth.  The commutator is left unevaluated.  (see also
normalorder(arg))

normalorder(arg)

arg - expression

normalorder(arg) uses the relation a.b  = b.a - com(b,a) to change
the order of operators in the expression, arg.  The preffered
order is determined by the user defined list X.  Every instance of
the first element of X, X[1], is commuted to the left, and so
forth.  Each commutator is evaluated.  (see also
Normalorder(arg)).
```

```
Simp(arg)

arg - expression

Simp(arg) combines the procedures, todot(arg), Normalorder(arg),
and distribute(arg) repeatedly to simplify expressions as much as
possible.

simp(arg)

arg - expression

simp(arg) combines the procedures, todot(arg), normalorder(arg),
and distribute(arg) repeatedly to simplify expressions as much as
possible.

todot(arg)

arg - expression

todot(arg) converts all instances of a carrot (^) operator to
repeated non-commutative multiplication.

tostar(arg)

arg - expression
```

```
tostar(arg) converts all instances of the noncommutative or dot
product `.` to the commutative or star product `*`.
```

B.1.3 Maple code

```
commutator:=module() export LoadAlg, Normalorder, normalorder,
tostar, Depend, todot, simp, Simp, dotProd, distribute, deg, com;

local _com4, _com3, _com2, _com1, _C, _evalcomconst, _comconst;

global `expand/Com`, `convert/.`;

_com4:=proc(A,B,dep) local answer; answer:=expand(Com(A,B));
answer:=subs(Com=_C,answer); answer:=eval(answer); answer; end proc;

_evalcomconst:=proc(f,g,s::set,dep,coms) local _ki, _kip, i, j,
coeff, f1, f2, g1, g2, answer, m1, m2, answer1, answer2, K, KP, b;
f1:=f; f2:=f; g1:=g; g2:=g; coeff:=1; _ki:=array(1..nops(dep));
_kip:=array(1..nops(dep)); m1:=array(1..nops(dep));
m2:=array(1..nops(dep)); for i from 1 to nops(dep) do
   _ki[i]:=0; _kip[i]:=0; m1[i]:=infinity; m2[i]:=infinity;
end do;
for j from 2 to nops(dep) do
   for i from 1 to j - 1 do
      if evalb(k[i,j] in s) then
         _ki[i]:=_ki[i] + k[i,j];
```

```
         _kip[j]:=_kip[j] + k[i,j];
         coeff:=coeff*(-coms[i,j])^(k[i,j])/k[i,j]!;
      end if;
   end do;
end do;
for i from 1 to nops(dep) do
   if _ki[i] <> 0 then
      if depends(f2,dep[i]) then
         if type(deg(f2,dep[i]),integer) then
            m2[i]:=deg(f2,dep[i]);
         end if;
         f2:=diff(f2,dep[i]$_ki[i]);
      else m2[i]:=0;
      end if;
      if depends(g1,dep[i]) then
         if type(deg(g1,dep[i]),integer) then
            m1[i]:=deg(g1,dep[i]);
         end if;
         g1:=diff(g1,dep[i]$_ki[i]);
      else m1[i]:=0;
      end if;
   end if;
   if _kip[i] <> 0 then
      if depends(f1,dep[i]) then
         if type(deg(f1,dep[i]),integer) then
            m1[i]:=deg(f1,dep[i]);
```

```
         end if;
         f1:=diff(f1,dep[i]$_kip[i]);
      else m1[i]:=0;
      end if;
      if depends(g2,dep[i]) then
         if deg(g2,dep[i]) <> FAIL and deg(g2,dep[i]) <> -infinity then
            m2[i]:=deg(g2,dep[i]);
         end if;
         g2:=diff(g2,dep[i]$_kip[i]);
      else m2[i]:=0;
      end if;
   end if;
end do; answer1:=coeff.(g1.f1); answer2:=-coeff.(f2.g2);
answer:=coeff.(g1.f1-f2.g2); b:=false; for i from 1 to nops(s) do
   K:=op(1,s[i]); KP:=op(2,s[i]);
   if min(m1[K],m1[KP]) < infinity or min(m2[KP],m2[K]) < infinity then
      b:=true;
      break;
   end if;
end do; if b then
   for i from 1 to nops(s) do
      K:=op(1,s[i]); KP:=op(2,s[i]);
      if min(m1[K],m1[KP]) < infinity then
         answer1:=sum(answer1,s[i]=1..min(m1[K],m1[KP]));
      else
         answer1:=Sum(answer1,s[i]=1..infinity);
```

```
      end if;
      if min(m2[KP],m2[K]) < infinity then
         answer2:=sum(answer2,s[i]=1..min(m2[KP],m2[K]));
      else
         answer2:=Sum(answer2,s[i]=1..infinity);
      end if;
   end do;
   answer:=answer1+answer2;
else
   for i from 1 to nops(s) do
      answer:=Sum(answer,s[i] = 1..infinity);
   end do;
end if; answer; end proc:

_comconst:=proc(f,g,dep) local i, j, lis, s, S,comp, f1, g1,
answer,_dep,_Coms; f1:=f; g1:=g; answer:=0; _dep:=convert(dep,list);
_Coms:=array(1..nops(_dep),1..nops(_dep),antisymmetric); for i from
1 to nops(_dep)-1 do
   for j from i+1 to nops(-dep) do
   _Coms[i,j]:=_C(op(i,_dep),op(j,_dep));
   end do;
end do; lis:={}; for j from 2 to nops(_dep) do
   for i from 1 to j-1 do
      if _Coms[i,j] <> 0 then
         lis:= lis union {k[i,j]};
      end if;
```

```
   end do
end do;
s:={}; S:=combinat[subsets](lis); while not S[finished] do
   s:= s union {S[nextvalue]()};
end do; s:=s minus {{}};
for i from 1 to nops(s) do
   comp:=lis  minus s[i];
   for j from 1 to nops(comp) do
      f1:=subs(comp[j] = 0, f1);
      g1:=subs(comp[j] = 0, g1);
   end do;
   answer:=answer + _evalcomconst(f1,g1,s[i],_dep,_Coms);
end do; answer; end proc:

_com3:=proc(A,B,dep) local
a,b,c,d,e,f,n1,n2,n3,n4,n5,n6,c1,c2,c3,_dep,term1,term2,
term3,term4,term5,term6,term7,term8,term9,term10,term11,
term12,term13,term14,answer; c1:=0; c2:=0; c3:=0;
_dep:=convert(dep,list); a:=op(1,_dep); if _C(a,op(3,_dep)) <> 0
then
   b:=op(3,dep);
elif _C(a,op(4,_dep)) <> 0 then
   b:=op(4,dep);
elif _C(a,op(5,_dep)) <> 0 then
   b:=op(5,dep);
elif _C(a,op(6,_dep)) <> 0 then
```

```
   b:=op(6,_dep);
else
   b:=op(2,_dep);
end if; _dep:=convert(convert(_dep,set) minus {a,b},list);
c:=op(1,_dep); if _C(c,op(3,_dep)) <> 0 then
   d:=op(3,_dep);
elif _C(c,op(4,_dep)) <> 0 then
   d:=op(4,_dep);
else
   d:=op(2,_dep);
end if; e:=op(1, (convert(_dep,set) minus {c,d}) ); f:=op(2,
(convert(_dep,set) minus {c,d}) ); c1:=_C(a,b); c2:=_C(c,d);
c3:=_C(e,f); n1:=subs(FAIL = infinity,{deg(B,a),deg(A,b)});
n1:=min( seq(op(i,n1),i=1..nops(n1)));\

n2:=subs(FAIL = infinity,{deg(A,a),deg(B,b)});  n2:=min(
seq(op(i,n2),i=1..nops(n2)));\

n3:=subs(FAIL = infinity,{deg(B,c),deg(A,d)});  n3:=min(
seq(op(i,n3),i=1..nops(n3)));\

n4:=subs(FAIL = infinity,{deg(A,c),deg(B,d)});  n4:=min(
seq(op(i,n4),i=1..nops(n4)));\

n5:=subs(FAIL = infinity,{deg(B,e),deg(A,f)});  n5:=min(
seq(op(i,n5),i=1..nops(n5)));\
```

```
n6:=subs(FAIL = infinity,{deg(A,e),deg(B,f)});  n6:=min(
seq(op(i,n6),i=1..nops(n6)));\


if n1 < infinity and n3 < infinity then
   term7:= sum( sum( (-c1)^k .(-c2)^l/k!/l! .
   diff( diff(B,a$k),c$l).diff( diff(A,b$k),d$l),k=1..n1),l=1..n3);\
elif n1 < infinity then
   term7:= Sum( sum( (-c1)^k .(-c2)^l/k!/l! .
   diff( diff(B,a$k),c$l).diff( diff(A,b$k),d$l),k=1..n1),l=1..n3);\
elif n3 < infinity then
   term7:= sum( Sum( (-c1)^k .(-c2)^l/k!/l! .
   diff( diff(B,a$k),c$l).diff( diff(A,b$k),d$l),k=1..n1),l=1..n3);\
else
   term7:= Sum( Sum( (-c1)^k .(-c2)^l/k!/l! .
   diff( diff(B,a$k),c$l).diff( diff(A,b$k),d$l),k=1..n1),l=1..n3);\
end if; if n2 < infinity and n4 < infinity then
   term8:= -sum( sum( (-c1)^k .(-c2)^l/k!/l! .
   diff( diff(A,a$k),c$l).diff( diff(B,b$k),d$l),k=1..n2),l=1..n4);\
elif n2 < infinity then
   term8:= -Sum( sum( (-c1)^k .(-c2)^l/k!/l! .
   diff( diff(A,a$k),c$l).diff( diff(B,b$k),d$l),k=1..n2),l=1..n4);\
elif n4 < infinity then
   term8:= -sum( Sum( (-c1)^k .(-c2)^l/k!/l! .
   diff( diff(A,a$k),c$l).diff( diff(B,b$k),d$l),k=1..n2),l=1..n4);\
```

```
else
   term8:= -Sum( Sum( (-c1)^k .(-c2)^l/k!/l! .
   diff( diff(A,a$k),c$l).diff( diff(B,b$k),d$l),k=1..n2),l=1..n4);\
end if;
if n1 < infinity and n5 < infinity then
   term9:= sum( sum( (-c1)^k .(-c3)^l/k!/l! .
   diff( diff(B,a$k),e$l).diff( diff(A,b$k),f$l),k=1..n1),l=1..n5);\
elif n1 < infinity then
   term9:= Sum( sum( (-c1)^k .(-c3)^l/k!/l! .
   diff( diff(B,a$k),e$l).diff( diff(A,b$k),f$l),k=1..n1),l=1..n5);\
elif n5 < infinity then
   term9:= sum( Sum( (-c1)^k .(-c3)^l/k!/l! .
   diff( diff(B,a$k),e$l).diff( diff(A,b$k),f$l),k=1..n1),l=1..n5);\
else
   term9:= Sum( Sum( (-c1)^k .(-c3)^l/k!/l! .
   diff( diff(B,a$k),e$l).diff( diff(A,b$k),f$l),k=1..n1),l=1..n5);\
end if;
if n2 < infinity and n6 < infinity then
   term10:= -sum( sum( (-c1)^k .(-c3)^l/k!/l! .
   diff( diff(A,a$k),e$l).diff( diff(B,b$k),f$l),k=1..n2),l=1..n6);\
elif n2 < infinity then
   term10:= -Sum( sum( (-c1)^k .(-c3)^l/k!/l! .
   diff( diff(A,a$k),e$l).diff( diff(B,b$k),f$l),k=1..n2),l=1..n6);\
elif n6 < infinity then
   term10:= -sum( Sum( (-c1)^k .(-c3)^l/k!/l! .
   diff( diff(A,a$k),e$l).diff( diff(B,b$k),f$l),k=1..n2),l=1..n6);\
```

```
else
   term10:= -Sum( Sum( (-c1)^k .(-c3)^l/k!/l! .
   diff( diff(A,a$k),e$l).diff( diff(B,b$k),f$l),k=1..n2),l=1..n6);\
end if;
if n3 < infinity and n5 < infinity then
   term11:= sum( sum( (-c2)^k .(-c3)^l/k!/l! .
   diff( diff(B,c$k),e$l).diff( diff(A,d$k),f$l),k=1..n3),l=1..n5);\
elif n1 < infinity then
   term11:= Sum( sum( (-c2)^k .(-c3)^l/k!/l! .
   diff( diff(B,c$k),e$l).diff( diff(A,d$k),f$l),k=1..n3),l=1..n5);\
elif n5 < infinity then
   term11:= sum( Sum( (-c2)^k .(-c3)^l/k!/l! .
   diff( diff(B,c$k),e$l).diff( diff(A,d$k),f$l),k=1..n3),l=1..n5);\
else
   term11:= Sum( Sum( (-c2)^k .(-c3)^l/k!/l! .
   diff( diff(B,c$k),e$l).diff( diff(A,d$k),f$l),k=1..n3),l=1..n5);\
end if;
if n4 < infinity and n6 < infinity then
   term12:= -sum( sum( (-c2)^k .(-c3)^l/k!/l! .
   diff( diff(A,c$k),e$l).diff( diff(B,d$k),f$l),k=1..n4),l=1..n6);\
elif n2 < infinity then
   term12:= -Sum( sum( (-c2)^k .(-c3)^l/k!/l! .
   diff( diff(A,c$k),e$l).diff( diff(B,d$k),f$l),k=1..n4),l=1..n6);\
elif n6 < infinity then
   term12:= -sum( Sum( (-c2)^k .(-c3)^l/k!/l! .
   diff( diff(A,c$k),e$l).diff( diff(B,d$k),f$l),k=1..n4),l=1..n6);\
```

```
else
   term12:= -Sum( Sum( (-c2)^k .(-c3)^l/k!/l! .
   diff( diff(A,c$k),e$l).diff( diff(B,d$k),f$l),k=1..n4),l=1..n6);\
end if;
if n1 < infinity and n3 < infinity and n5 < infinity then
  term13:=sum( sum( sum( (-c1)^k . (-c2)^l . (-c3)^m/k!/l!/m! .
  diff( diff( diff( B,a$k),c$l),e$m). diff( diff( diff( A,b$k),d$l),f$m)\
             ,m=1..n5),l=1..n3),k=1..n1);
elif n1 < infinity and n3 < infinity then
  term13:=sum( sum( Sum( (-c1)^k . (-c2)^l .(-c3)^m/k!/l!/m! .
  diff( diff( diff( B,a$k),c$l),e$m). diff( diff( diff( A,b$k),d$l),f$m)\
             ,m=1..n5),l=1...n3),k=1..n1);
elif n1 < infinity and n5 < infinity then
  term13:=sum( Sum( sum( (-c1)^k . (-c2)^l .(-c3)^m/k!/l!/m! .
  diff( diff( diff( B,a$k),c$l),e$m). diff( diff( diff( A,b$k),d$l),f$m)\
             ,m=1..n5),l=1...n3),k=1..n1);
elif n3 < infinity and n5 < infinity then
  term13:=Sum( sum( sum( (-c1)^k . (-c2)^l .(-c3)^m/k!/l!/m! .
  diff( diff( diff( B,a$k),c$l),e$m). diff( diff( diff( A,b$k),d$l),f$m)\
             ,m=1..n5),l=1...n3),k=1..n1);
elif n1 < infinity then
  term13:=sum( Sum( Sum( (-c1)^k . (-c2)^l .(-c3)^m/k!/l!/m! .
  diff( diff( diff( B,a$k),c$l),e$m). diff( diff( diff( A,b$k),d$l),f$m)\
             ,m=1..n5),l=1...n3),k=1..n1);
elif n3 < infinity then
  term13:=Sum( sum( Sum( (-c1)^k . (-c2)^l .(-c3)^m/k!/l!/m! .
```

```
  diff( diff( diff( B,a$k),c$l),e$m). diff( diff( diff( A,b$k),d$l),f$m)\
              ,m=1..n5),l=1...n3),k=1..n1);
elif n5 < infinity then
  term13:=Sum( Sum( sum( (-c1)^k . (-c2)^l .(-c3)^m/k!/l!/m! .
  diff( diff( diff( B,a$k),c$l),e$m). diff( diff( diff( A,b$k),d$l),f$m)\
              ,m=1..n5),l=1...n3),k=1..n1);
else
  term13:=Sum( Sum( Sum( (-c1)^k . (-c2)^l .(-c3)^m/k!/l!/m! .
  diff( diff( diff( B,a$k),c$l),e$m). diff( diff( diff( A,b$k),d$l),f$m)\
              ,m=1..n5),l=1...n3),k=1..n1);
end if;
if n2 < infinity and n4 < infinity and n6 < infinity then
  term14:=-sum( sum( sum( (-c1)^k . (-c2)^l . (-c3)^m/k!/l!/m! .
  diff( diff( diff( A,a$k),c$l),e$m). diff( diff( diff( B,b$k),d$l),f$m)\
              ,m=1..n6),l=1..n4),k=1..n2);
elif n2 < infinity and n4 < infinity then
  term14:=-sum( sum( Sum( (-c1)^k . (-c2)^l . (-c3)^m/k!/l!/m! .
  diff( diff( diff( A,a$k),c$l),e$m). diff( diff( diff( B,b$k),d$l),f$m)\
              ,m=1..n6),l=1..n4),k=1..n2);
elif n2 < infinity and n6 < infinity then
  term14:=-sum( Sum( sum( (-c1)^k . (-c2)^l . (-c3)^m/k!/l!/m! .
  diff( diff( diff( A,a$k),c$l),e$m). diff( diff( diff( B,b$k),d$l),f$m)\
              ,m=1..n6),l=1..n4),k=1..n2);
elif n4 < infinity and n6 < infinity then
  term14:=-Sum( sum( sum( (-c1)^k . (-c2)^l . (-c3)^m/k!/l!/m! .
  diff( diff( diff( A,a$k),c$l),e$m). diff( diff( diff( B,b$k),d$l),f$m)\
```

```
             ,m=1..n6),l=1..n4),k=1..n2);
elif n2 < infinity then
  term14:=-sum( Sum( Sum( (-c1)^k . (-c2)^l . (-c3)^m/k!/l!/m! .
  diff( diff( diff( A,a$k),c$l),e$m). diff( diff( diff( B,b$k),d$l),f$m)\
             ,m=1..n6),l=1..n4),k=1..n2);
elif n4 < infinity then
  term14:=-Sum( sum( Sum( (-c1)^k . (-c2)^l . (-c3)^m/k!/l!/m! .
  diff( diff( diff( A,a$k),c$l),e$m). diff( diff( diff( B,b$k),d$l),f$m)\
             ,m=1..n6),l=1..n4),k=1..n2);
elif n6 < infinity then
  term14:=-Sum( Sum( sum( (-c1)^k . (-c2)^l . (-c3)^m/k!/l!/m! .
  diff( diff( diff( A,a$k),c$l),e$m). diff( diff( diff( B,b$k),d$l),f$m)\
             ,m=1..n6),l=1..n4),k=1..n2);
else
  term14:=-Sum( Sum( Sum( (-c1)^k . (-c2)^l . (-c3)^m/k!/l!/m! .
  diff( diff( diff( A,a$k),c$l),e$m). diff( diff( diff( B,b$k),d$l),f$m)\
             ,m=1..n6),l=1..n4),k=1..n2);
end if; if n1 = 0 then
   term1:=0; term7:=0; term9:=0; term13:=0;
elif n1=infinity then
   term1:=Sum((-c1)^k/k!.diff(B,a$k).diff(A,b$k),k=1..n1);
else
   term1:=sum((-c1)^k/k!.diff(B,a$k).diff(A,b$k),k=1..n1);
end if;
if n2 = 0 then
   term2:=0; term8:=0; term10:=0; term14:=0;
```

```
elif n2=infinity then
   term2:=-Sum((-c1)^k/k!.diff(A,a$k).diff(B,b$k),k=1..n2);
else
   term2:=-sum((-c1)^k/k!.diff(A,a$k).diff(B,b$k),k=1..n2);
end if;
if n3=0 then
   term3:=0; term7:=0; term11:=0; term13:=0;
elif n3=infinity then
   term3:=Sum((-c2)^k/k!.diff(B,c$k).diff(A,d$k),k=1..n3);
else
   term3:=sum((-c2)^k/k!.diff(B,c$k).diff(A,d$k),k=1..n3);
end if;
if n4=0 then
   term4:=0; term8:=0; term12:=0; term14:=0;
elif n4=infinity then
   term4:=-Sum((-c2)^k/k!.diff(A,c$k).diff(B,d$k),k=1..n4);
else
   term4:=-sum((-c2)^k/k!.diff(A,c$k).diff(B,d$k),k=1..n4);
end if;
if n5=0 then
   term5:=0; term9:=0; term11:=0; term13:=0;
elif n5=infinity then
   term5:=Sum((-c3)^k/k!.diff(B,e$k).diff(A,f$k),k=1..n5);
else
   term5:=sum((-c3)^k/k!.diff(B,e$k).diff(A,f$k),k=1..n5);
end if;
```

```
if n6=0 then
   term6:=0; term10:=0; term12:=0; term14:=0;
elif n6=infinity then
   term6:=-Sum((-c3)^k/k!.diff(A,e$k).diff(B,f$k),k=1..n6);
else
   term6:=-sum((-c3)^k/k!.diff(A,e$k).diff(B,f$k),k=1..n6);
end if;
answer:=term1+term2+term3+term4+term5+term6+term7+term8+
        term9+term10+term11+term12+term13+term14;\
answer; end proc:

_com2:=proc(A,B,dep) local n1, n2, n3, n4,
answer,a,b,c,d,c1,c2,term1, term2, term3, term4, term5, term6;
c1:=0; c2:=0; a:=op(1,dep); if _C(a,op(3,dep)) <> 0 then
   b:=op(3,dep);
elif _C(a,op(4,dep)) <> 0 then
   b:=op(4,dep);
else
   b:=op(2,dep);
end if; c:=op(1, convert(dep,set) minus {a,b}); d:=op(2,
convert(dep,set) minus {a,b}); c1:=_C(a,b); c2:=_C(c,d);
n1:=subs(FAIL = infinity, {deg(B,a),deg(A,b)});  n1:=min(
seq(op(i,n1),i=1..nops(n1))); n2:=subs(FAIL = infinity,
{deg(A,a),deg(B,b)});  n2:=min( seq(op(i,n2),i=1..nops(n2)));
n3:=subs(FAIL = infinity, {deg(B,c),deg(A,d)});  n3:=min(
seq(op(i,n3),i=1..nops(n3))); n4:=subs(FAIL = infinity,
```

```
{deg(A,c),deg(B,d)});  n4:=min( seq(op(i,n4),i=1..nops(n4)));


if n1 < infinity and n3 < infinity then
   term5:= sum( sum( (-c1)^k .(-c2)^l/k!/l! .
   diff( diff(B,a$k),c$l).diff( diff(A,b$k),d$l),k=1..n1),l=1..n3);
elif n1 < infinity then
   term5:= Sum( sum( (-c1)^k .(-c2)^l/k!/l! .
   diff( diff(B,a$k),c$l).diff( diff(A,b$k),d$l),k=1..n1),l=1..n3);
elif n3 < infinity then
   term5:= sum( Sum( (-c1)^k .(-c2)^l/k!/l! .
   diff( diff(B,a$k),c$l).diff( diff(A,b$k),d$l),k=1..n1),l=1..n3);
else
   term5:= Sum( Sum( (-c1)^k .(-c2)^l/k!/l! .
   diff( diff(B,a$k),c$l).diff( diff(A,b$k),d$l),k=1..n1),l=1..n3);
end if;
if n2 < infinity and n4 < infinity then
   term6:=- sum( sum( (-c1)^k .(-c2)^l/k!/l! .
   diff( diff(A,a$k),c$l).diff( diff(B,b$k),d$l),k=1..n1),l=1..n3);
elif n1 < infinity then
   term6:= -Sum( sum( (-c1)^k .(-c2)^l/k!/l! .
   diff( diff(A,a$k),c$l).diff( diff(B,b$k),d$l),k=1..n1),l=1..n3);
elif n3 < infinity then
   term6:= -sum( Sum( (-c1)^k .(-c2)^l/k!/l! .
   diff( diff(A,a$k),c$l).diff( diff(B,b$k),d$l),k=1..n1),l=1..n3);
else
   term6:= -Sum( Sum( (-c1)^k .(-c2)^l/k!/l! .
```

```
   diff( diff(A,a$k),c$l).diff( diff(B,b$k),d$l),k=1..n1),l=1..n3);
end if;
if n1 = 0 then
   term1:=0; term5:=0;
elif n1 = infinity then
   term1:=Sum((-c1)^k/k!.diff(B,a$k).diff(A,b$k),k=1..n1);
else
   term1:=sum((-c1)^k/k!.diff(B,a$k).diff(A,b$k),k=1..n1);
end if;
if n2 = 0 then
   term2:=0; term6:=0;
elif n2 = infinity then
   term2:=-Sum((-c1)^k/k!.diff(A,a$k).diff(B,b$k),k=1..n2);
else
   term2:=-sum((-c1)^k/k!.diff(A,a$k).diff(B,b$k),k=1..n2);
end if;
if n3 = 0 then
   term3:=0; term5:=0;
elif n3 = infinity then
   term3:=Sum((-c2)^k/k!.diff(B,c$k).diff(A,d$k),k=1..n3);
else
   term3:=sum((-c2)^k/k!.diff(B,c$k).diff(A,d$k),k=1..n3);
end if;

if n4 = 0 then
   term4:=0; term6:=0;
```

```
elif n4 = infinity then
   term4:=-Sum((-c2)^k/k!.diff(A,c$k).diff(B,d$k),k=1..n4);
else
   term4:=-sum((-c2)^k/k!.diff(A,c$k).diff(B,d$k),k=1..n4);
end if; answer:=term1 + term2 + term3 + term4 + term5 + term6;
answer; end proc:

_com1:=proc(A,B,dep) local c,n1,n2,answer,a,b; a:=op(1,dep);
b:=op(2,dep); n1:=subs(FAIL = infinity,{deg(A,b),deg(B,a)});
n2:={deg(A,a),deg(B,b)} minus {FAIL,-infinity}; n1:=min(
seq(op(i,n1),i=1..nops(n1))); n2:=min( seq(op(i,n2),i=1..nops(n2)));
c:=_C(a,b); if n1 = infinity and n2 = infinity then
   answer:=Sum((-c)^k/k! . (diff(B,a$k).diff(A,b$k) -
   diff(A,a$k).diff(B,b$k)),k=1..infinity);
elif n1 = infinity then
   answer:=Sum((-c)^k/k! . diff(B,a$k).diff(A,b$k),k=1..n1) -
   sum((-c)^k/k! .diff(A,a$k).diff(B,b$k),k=1..n2);
elif n2 = infinity then
   answer:=sum((-c)^k/k! . diff(B,a$k).diff(A,b$k),k=1..n1) -
   Sum((-c)^k/k! .diff(A,a$k).diff(B,b$k),k=1..n2);
else
   answer:=sum((-c)^k/k! . diff(B,a$k).diff(A,b$k),k=1..n1) -
   sum((-c)^k/k! .diff(A,a$k).diff(B,b$k),k=1..n2);
end if; answer; end proc:
_C:=proc(A,B) local a, b, i; for i from
1 to nops(X) do
```

```
   if A = X[i] then a:=i;  end if;
   if B = X[i] then b:=i; end if;
   i;
end do; if type(a,integer) and type(b,integer) then
   C[a,b];
else
   0;
end if;
end proc;

LoadAlg:=proc() global X, C, _ALG; local i, j; if [args] = [] then
   _ALG:=0;
else _ALG:=args; end if; if _ALG = 1 then
   unassign('X'); unassign('C');
   X:=[x1,x2]; C:=array(1..2,1..2,antisymmetric); C[1,2]:=C12;
elif _ALG = 2 then
   unassign('X'); unassign('C');
   X:=[x1,x2,x3,x4]; C:=array(1..4,1..4,antisymmetric);
   C[1,3]:=0; C[1,4]:=0; C[2,3]:=0; C[2,4]:=0;
   C[1,2]:=C12; C[3,4]:=C34;
elif _ALG = 3 then
   unassign('X');  unassign('C');
   X:=[x1,x2,x3,x4,x5,x6]; C:=array(1..6,1..6,antisymmetric);
   C[1,3]:=0; C[1,4]:=0; C[1,5]:=0;  C[1,6]:=0;
   C[2,3]:=0; C[2,4]:=0; C[2,5]:=0;  C[2,6]:=0;
   C[3,5]:=0; C[3,6]:=0;
```

```
   C[4,5]:=0; C[4,6]:=0;
   C[1,2]:=C12; C[3,4]:=C34; C[5,6]:=C56;
elif _ALG = 4 then
   unassign('X');  unassign('C');
   X:=[Lx,Ly,Lz];  C:=array(1..3,1..3,antisymmetric);
   C[1,2]:=ih.Lz;  C[1,3]:=-ih.Ly;  C[2,3]:=ih.Lx;
end if; end proc:

Normalorder:=proc(context) local new, cntxt, p, q, i, j, b;
cntxt:=eval(context); new:=cntxt; if nops(new)>1 and
type(new,constant) = false and type(new,indexed) = false then
   new:=map(Normalorder,new);
end if; if type(new,`.`) and nops(Depend(new)) >= 1 then
   for i from 1 to nops(new)-1 do
      p:=0; q:=0;
      for j from 1 to nops(X) do
         if depends(op(i,new),X[j]) then p:=j; end if;
         if depends(op(i+1,new), X[j]) then q:=j end if;
      end do;
      if q < p then
         new:=Normalorder((subsop(i = op(i+1,new).op(i,new) -
         Normalorder((Com(op(i+1,new),op(i,new)))),i+1 = 1, new)));
         break;
      end if;
    end do;
end if; eval(new); end proc:
```

```
normalorder:=proc(context) local new, cntxt, p, q, i, j, b;
cntxt:=eval(context); new:=cntxt; if nops(new)>1 and
type(new,constant) = false and type(new,indexed) = false then
   new:=map(normalorder,new);
end if; if type(new,`.`) and nops(Depend(new)) >= 1 then
   for i from 1 to nops(new)-1 do
      p:=0; q:=0;
      for j from 1 to nops(X) do
         if depends(op(i,new),X[j]) then p:=j; end if;
         if depends(op(i+1,new), X[j]) then q:=j end if;
      end do;
      if q < p then
         new:=normalorder((subsop(i= op(i+1,new).op(i,new) -
         normalorder(com(op(i+1,new),op(i,new))),i+1 = 1, new)));
         break;
      end if;
    end do;
end if; eval(new); end proc:

tostar:=proc(context) local new; new:=context; if type(context,`.`)
then
   new:=convert(context,`*`);
   new:=map(tostar,new);
elif nops(context)>1 and type(context,constant) = false and
type(context,indexed) = false then
```

```
   new:=map(tostar,context);
else new:=context; end if; new; end proc:

Depend:=proc(f) local d,i,o; o:=convert(X,set); d:={}; for i from 1
to nops(X) do
   if depends(f,o[i]) then
      d:= d union {o[i]}
   end if;
end do; d; end proc:

todot:=proc(context) local new; new:=context; if type(context,`^`)
and type(op(2,context),integer) then
   if op(2,context) > 0 then
      new:=convert(new,`.`);
   end if;
end if; if (nops(context)>1 or nops(op(1,convert(context,list))) > 1
) and type(context,indexed) = false and type(context, constant) =
false then
   new:= map(todot,new);
end if; new; end proc:


simp:=proc(context) local new;
new:=eval(normalorder(distribute(todot(context)))); if new<>context
then
   new:=simp(new);
```

```
end if; simplify(new); end proc:

Simp:=proc(context) local new;
new:=eval(Normalorder(distribute(todot(context)))); if new<>context
then
   new:=Simp(new);
end if; simplify(new); end proc:

`expand/Com`:=proc(A,B) local new,l, i,i1,i2,temp1, temp2,a,b;
a:=expand(A); b:=expand(B); ##if A is a +, then we turn it into a
sum a Commutators if type(a,`+`) then
   new:=0;
   l:=convert(a,list);
   for i from 1 to nops(l) do
   ##This loop adds up the commutators to make the sum of Commutators
      new:=new+expand(Com(l[i],b));
   end do;
##If B is a + we turn it into a sum of Commutators elif
type(b,`+`) then
   new:=0;
   l:=convert(b,list);
   for i from 1 to nops(l) do
      new:=new+expand(Com(a,l[i]));
   end do;
##If A is a . then we expand it using Leibniz's rule. elif
type(a,`.`) then
```

```
   l:=convert(a,list);
   temp1:=l[1];                ##temp1 is the first term in the .
   temp2:=1;                   ##temp2 is the remaining terms in the .
   for i from 2 to nops(l) do
      temp2:=temp2 . l[i];
   end do;
   new:=temp1 . expand(Com(temp2,b)) + expand(Com(temp1,b)).temp2;
   ##Leibniz's rule
##If B is a . then we expand it using Leibniz's rule. elif
type(b,`.`) then
   l:=convert(b,list);
   temp1:=l[1];
   temp2:=1;
   for i from 2 to nops(l) do
      temp2:=temp2 . l[i];
   end do;
   new:=temp1 . expand(Com(a,temp2)) + expand(Com(a,temp1)) . temp2;
##If A is ^ then is expanded also using Leibniz's rule, where p^2
= p.p elif type(a,`^`) then
   l:=convert(a,list);
   if type(l[2],constant) and l[2]>0 then
      new:=l[1] . expand(Com(l[1]^(l[2] - 1),b)) +
      expand(Com(l[1],b)).l[1]^(l[2]-1);
   elif type(l[2],constant) and l[2]<0 then
      new:='Com(a,b)';
      ##If the power is negative, then we don't do anything.
```

```
   else new:=Sum( l[1]^(j-1) . expand(Com(l[1],b)) .
             l[1]^(l[2]-j),j=1..l[2]);
   end if;
##If B is a ^ then it is expanded using Leibniz's rule where p^n =
p.p. .. p elif type(b,`^`) then
   l:=convert(b,list);
   if type(l[2],constant)and l[2]>0 then
      new:=l[1] . expand(Com(a,l[1]^(l[2]-1))) +
           expand(Com(a,l[1])).l[1]^(l[2]-1);
   elif type(l[2],constant) and l[2] < 0 then
      new:='Com(a,b)';
   else new:=Sum(l[1]^(j-1) . expand(Com(a,l[1])) .
             l[1]^(l[2]-j),j=1..l[2]);
   end if;
##If A is a *, then all of its operators must be a c number except
for one.  We use an expansion similar to the Liebniz's rule.  When
evaluated the result will have factored out all of the c numbers.
elif type(a,`*`) then
   l:=convert(a,list);
   temp1:=1;
   for i from 2 to nops(l) do
      temp1:=temp1*l[i];
   end do;
   new:=l[1]*expand(Com(temp1,b)) + temp1*expand(Com(l[1],b));
##If B is a * then all of its operators must be a c number except
for one.  We use an expansion similar to the Liebniz's rule.  When
```

```
evaluated the result will have factored out all of the c numbers.
elif type(b,‘*‘) then
   l:=convert(b,list);
   temp1:=1;
   for i from 2 to nops(l) do
      temp1:=temp1*l[i];
   end do;
   new:=l[1]*expand(Com(a,temp1)) + temp1*expand(Com(a,l[1]));
##If A is a greek sum, then we convert it to be a greek sum of
Commutators. elif type(a,specfunc(anything,Sum)) or
type(a,specfunc(anything,sum)) then
   l:=convert(a,list);
   new:=Sum(expand(Com(l[1],b)),l[2]);
##If B is a greek sum, then we convert it to be a greek sum of
commutators. elif type(b,specfunc(anything,Sum)) or
type(b,specfunc(anything,sum)) then
  l:=convert(b,list);
  new:=Sum(expand(Com(a,l[1])),l[2]);
##If A is a DotProd function (defined below), then we use a
formula to expand it in terms of fundamental comms. elif
type(a,specfunc(anything, dotProd)) or type(a,specfunc(anything,
DotProd)) then
   l:=convert(a,list);
   i:=convert(l[2],list)[1];
   i1:=convert(convert(l[2],list)[2],list)[1];
   i2:=convert(convert(l[2],list)[2],list)[2];
```

```
   new:=eval(Sum( DotProd(l[1],i=i1..j-1) .
   expand(subs(i=j,Com(l[1],b))) . DotProd(l[1],i = j+1..i2), j = i1..i2));
##If B is a DotPord function then we use a formula to expand it in
terms of fundamental comms. elif type(b,specfunc(anything,
dotProd)) or type(b,specfunc(anything, DotProd)) then
   l:=convert(b,list);
   i:=convert(l[2],list)[1];
   i1:=convert(convert(l[2],list)[2],list)[1];
   i2:=convert(convert(l[2],list)[2],list)[2];
   new:=eval(Sum( DotProd(l[1],i=i1..j-1) .
   expand(subs(i=j,Com(a,l[1]))) . DotProd(l[1],i = j+1..i2), j = i1..i2));
##Finally, the Com is left unevaluted. else new:=Com(a,b); end if;
new; end proc: dotProd:=proc(f,e) ##This procedure works like
product(a[i],i=1..n) except using the non-commutative product
instead of the '*' product. local k,m,n,i,answer;
k:=convert(e,list)[1]; m:=convert(convert(e,list)[2],list)[1];
n:=convert(convert(e,list)[2],list)[2]; answer:=1; if
type(m,integer) and type(n,integer) then
  for i from m by 1 while i <=n do
    answer:=answer.subs(k=i,f);
  end do;
else answer:='dotProd(f,e)'; end if; answer;
end proc:

distribute:=proc(context) local cntxt, b, new, i, j, sumelem;
cntxt:=eval(context); new:=cntxt; if (nops(new) > 1 or
```

```
nops(op(1,convert(new,list))) > 1) and (type(new,constant) = false
and type(new,indexed) = false) then
   new:=map(distribute,new);
end if; if type(new,`.`) then
   for i from 1 to nops(new) do  ##Distribution over addition
      if type(op(i,new),`+`) then
      ##If a type `+` is found as an element of a non-commutative product
         sumelem:=op(i,new);
         new:=distribute(sum(subsop(i=op(j,sumelem),cntxt),j=1..nops(sumelem)))
         break;
      end if;
   end do;
end if; eval(new); end proc:

deg:=proc(a,x) local deg; deg:=degree(tostar(a),x); deg; end proc:
`convert/.`:=proc(context)
    ##converts 'context' to an object of type `.`
    ##if 'context' is a type `^` object, then it is
    ##treated as though `^` implied the non-commutative product.
    ##For example, (a+b)^2 becomes (a+b).(a+b).
    ##However, any other type, each element is treated as a
    ##factor in a non-commutative product.
    ##For example, a+b becomes a.b.
local new, n, i; if type(context,list) then  ##If context is a list,
each element becomes a factor in a non-commutative product
   new:=1;
```

```
   for i from 1 to nops(context) do
      new:=new.op(i,context);
   end do;
elif type(context,`^`) then  ##If context is a `^`, the base is
multiplied by itself according to normal power rules using the
noncommutative product
   new:=convert(context,list)[1]; n:=convert(context,list)[2];
   new:=dotProd(new,i=1..n);
else new:=convert(convert(context,list),`.`); ##If context is a
arbitrary type, it is converted a list and then converted to a
non-commutative product end if; new; end proc:

com:=proc(A,B) local dep, coms, answer, i, j,x; x:=convert(X,set);
if _ALG = 1 then
   answer:=_com1(A,B,X);
elif _ALG = 2 then
   answer:=_com2(A,B,X);
elif _ALG = 3 then
   answer:=_com3(A,B,X);
elif _ALG = 4 then
   answer:=_com4(A,B,X);
else
   dep:=convert(Depend(A) union Depend(B),list);
   coms:={};
   for i from 1 to nops(dep)-1 do
      for j from i + 1 to nops(dep) do
```

```
         coms:=coms union Depend(_C(dep[i],dep[j]));
      end do;
   end do;
coms;
   if nops(coms intersect x) = 0 then
      answer:=_comconst(A,B,dep);
   else
      answer:=_com4(A,B,dep);
   end if;
end if; answer; end proc:

end module;
```

B.2 One-dimensional perturbation theory algorithms

In addition to the algorithms for the manipulation of noncommuative expressions developed in the previous section, I also developed algorithms for evaluating expressions of the form $< n|V|m >$, where $|n >$ represents the eigenstates of the unpertubred SHO. In addition, algorithms were implemented to evaluate the formulas that arise in perturbation theory. This section gives the Maple algorithms for these procedures.

B.2.1 Algorithms, syntax, and descriptions

There are two algorithms that are expected to be called by the user, “Bracket” and “bracket”. These two algorithms each call a subroutine called “Braket” and “braket” respectively.

```
Bracket(bra,V,ket)

bra - nonnegative integer or expression

V - expression.  It is assumed that this expression has been
evaluated using the sequence tostar(simp(V)) from the commutator
module.  Furthermore, it is assumed that the raising operator is
given by ‘d‘ and the lowering operator is given by ‘a‘ and that in
the commutator module, X = [d,a].

ket - nonnegative integer or expression

Bracket(bra,V,ket) evaluates the the expression <bra|V|ket> and
leaves the resulting expression in terms of Kroniker deltas.

bracket(bra,V,ket)

bra - nonnegative integer or expression

V - expression.  It is assumed that this expression has been
evaluated using the sequence tostar(simp(V)) from the commutator
module.  Furthermore, it is assumed that the raising operator is
given by ‘d‘ and the lowering operator is given by ‘a‘ and that in
the commutator module, X = [d,a].

 ket - nonnegative integer or expression
```

bracket(bra,V,ket) evaluates the the expression <bra|V|ket> and evaluates all Kroniker deltas.

B.2.2 Maple code

```
Bracket:=proc(N,V,M)
local new,v;
v:=expand(V);
if type(N,list)
and type(M,list) then
  if nops(N) <> nops(M) then
      new:=FAIL;
   elif nops(N) = 3 then
      new:=Braket(op(1,N),v,op(1,M));
      new:=Braket(op(2,N),subs(b=a,e=d,new),op(2,M));
      new:=Braket(op(3,N),subs(c=a,f=d,new),op(3,M));
   elif nops(N) = 2 then
      new:=Braket(op(1,N),v,op(1,M));
      new:=Braket(op(2,N),subs(b=a,e=d,new),op(2,M));
   else
       new:=FAIL;
   end if;
elif type(N,list) xor type(M,list) then
   new:=FAIL;
else
```

```
   new:=Braket(N,v,M);
end if;
new;
end proc;

bracket:=proc(N,V,M)
local new,v;
v:=expand(V);
if type(N,list)
and type(M,list) then
  if nops(N) <> nops(M) then
      new:=FAIL;
   elif nops(N) = 3 then
      new:=braket(op(1,N),v,op(1,M));
      new:=braket(op(2,N),subs(b=a,e=d,new),op(2,M));
      new:=braket(op(3,N),subs(c=a,f=d,new),op(3,M));
   elif nops(N) = 2 then
      new:=braket(op(1,N),v,op(1,M));
      new:=braket(op(2,N),subs(b=a,e=d,new),op(2,M));
   else
       new:=FAIL;
   end if;
elif type(N,list) xor type(M,list) then
   new:=FAIL;
else
   new:=braket(N,v,M);
```

```
end if;
new;
end proc;

Braket:=proc(N,V,M) local new,D,A,n,m,i,j;
if type(V,`+`) then
   new:=0;
   for i from 1 to nops(V) do
      new:=new + Braket(N,op(i,V),M);
   end do;
else
   D:=degree(V,d); A:=degree(V,a);
   new:=subs(d=1,a=1,V);
   n:=N-D; m:=M-A;
   new:=simplify(new*sqrt(product(N-i,i=0..D-1)*
   product(M-j,j=0..A-1))*delta[n,m]) assuming M > A and N > D;
end if;
new;
end proc;

braket:=proc(N,V,M)
local new,D,A,n,m,i,Delta,j;
   Delta:=proc(q,p)
   local answer;
   if type(p-q, constant) then
      if q = p then
```

```
         answer:=1;
      else
         answer:=0;
      end if;
   else
      answer:='Delta(q,p)';
   end if;
   answer;
   end proc;
if type(V,`+`) then
   new:=0;
   for i from 1 to nops(V) do
      new:=new + braket(N,op(i,V),M);
   end do;
else
   D:=degree(V,d); A:=degree(V,a);
   new:=subs(d=1,a=1,V);
   n:=N-D; m:=M-A;
   new:=simplify(new*sqrt(product(N-i,i=0..D-1)*
   product(M-j,j=0..A-1))*Delta(n,m)) assuming M > A and N > D;
end if; new; end proc;
```

The following code uses the procedures given above to evaluate the Perturbation corrections to fourth order.

```
En:=n->h*omega*(n+1/2); E:=(i,j)->En(i) - En(j);  ##The nth
```

```
unperturbed energy and the difference between unperturbed energies

sump:=proc(exprsn,i,n,range) ##sump acts like sum(), but skips the
##nth term that does not appear in the sums in perturbation
\index{perturbation theory} theory local new;
new:=sum(exprsn,i=n-range...n-1) + sum(exprsn,i=n+1..n+range);
new; end proc;

V:=(n,j)->bracket(n,P,j);  ##V(n,j) is the matrix element <n|V|j>.
P is the perturbation \index{perturbation theory} term defined
previously

for i from 1 to 3 do;

convert( series( map(factor,collect(simplify(

V(n,n) +  ##First Order

sump(V(j,n)*V(n,j)/E(n,j),j,n,20) +  ##Second Order

sump(sump(V(n,k)*V(k,j)*V(j,n)/E(k,n)/E(j,n),k,n,20),j,n,20) -
V(n,n)*sump(V(j,n)*V(n,j)/E(j,n)^2,j,n,20)  ##Third Order

+
sump(V(n,j)*V(j,n)/E(j,n),j,n,20)*sump(V(n,j)*V(j,n)/E(j,n)^2,j,n,20)
- V(n,n)^2*sump(V(n,j)*V(j,n)/E(j,n)^3,j,n,20) +
```

```
V(n,n)*sump(sump((1/E(l,n)^2/E(j,n) +
1/E(l,n)/E(j,n)^2)*V(n,l)*V(l,j)*V(j,n),j,n,20),l,n,20) -
sump(sump(sump(
                V(n,l)*V(l,j)*V(j,k)*V(k,n)/E(l,n)/E(j,n)/E(k,n)
                ,k,n,20),j,n,20),l,n,20)
##Fourth order
),alpha)),alpha=0,5),polynom);

end do;
```

B.3 Two-dimensional perturbation theory algorithms

For two-dimensional perturbation theory, the "Bracket" and "bracket" procedures are replaced by procedures of the same name but that have been altered to handle multidimensional inputs. The code that I implemented had the potential to calculate three-dimensional brackets, although I never actually did three-dimensional calculations in Snyder space. These new algorithms still call "Braket" and "braket" as defined in one-dimension, so these procedures must be defined in the worksheet. The following sections describe the new syntax for these algorithms, and gives the specific implementation.

B.3.1 Algorithms, syntax and descriptions

```
Bracket([bra1,bra2, bra3],V,[ket1,ket2,ket3])
```

```
bra1 - nonnegative integer or expression

bra2 - nonnegative integer or expression (optional)

bra3 - nonnegative integer or expression (optional)

V - expression.  It is assumed that this expression has been
evaluated using the sequence tostar(simp(V)) from the commutator
module.  Furthermore, it is assumed that the raising operator and
lowering operator in the x-dimension is given by ‘d‘ and ‘a‘
respectively, that the raising and lowering operators in the
y-dimension is ‘e‘ and ‘b‘ respectively, and that the raising and
lowering operators in the z-dimension  and the lowering operator
is given by ‘a‘ and that in the commutator module, X = [d,a].

ket - nonnegative integer or expression

Bkt([bra1,bra2],V,[ket1,ket2]) evaluates the the expression
<bra|V|ket> and leaves the resulting expression in terms of
Kroniker deltas.

bracket([bra1,bra2],V,[ket1,ket2])

bra1 - nonnegative integer or expression
```

```
bra2 - nonnegative integer or expression (optional)

bra3 - nonnegative integer or expression (optional)


V - expression.  It is assumed that this expression has been
evaluated using the sequence tostar(simp(V)) from the commutator
module.  Furthermore, it is assumed that the raising operator and
lowering operators in the x-dimension are given by ‘d‘ and  ‘a‘
respectively and that the raising and lowering operators in the
y-dimension are given by ‘e‘ and ‘b‘ respectively, and that ‘d‘ is
ordered to the left of ‘a‘ and that ‘e‘ is ordered to the left of
‘b‘1 in the commutator module.

ket1 - nonnegative integer or expression

ket2 - nonnegative integer or expression

bkt([bra1,bra2],V,[ket1,ket2]) evaluates the the expression
<bra|V|ket> and evaluates all Kroniker deltas.
```

B.3.2 Maple code

```
Bracket:=proc(N,V,M)
local new,v;
v:=expand(V); if type(N,list)
```

```
and type(M,list) then
  if nops(N) <> nops(M) then
      new:=FAIL;
   elif nops(N) = 3 then
      new:=Braket(op(1,N),v,op(1,M));
      new:=Braket(op(2,N),subs(b=a,e=d,new),op(2,M));
      new:=Braket(op(3,N),subs(c=a,f=d,new),op(3,M));
   elif nops(N) = 2 then
      new:=Braket(op(1,N),v,op(1,M));
      new:=Braket(op(2,N),subs(b=a,e=d,new),op(2,M));
   else
       new:=FAIL;
   end if;
elif type(N,list) xor type(M,list) then
   new:=FAIL;
else
   new:=Braket(N,v,M);
end if;
new;
end proc;

bracket:=proc(N,V,M)
local new,v;
v:=expand(V);
if type(N,list)
and type(M,list) then
```

```
  if nops(N) <> nops(M) then
      new:=FAIL;
   elif nops(N) = 3 then
      new:=braket(op(1,N),v,op(1,M));
      new:=braket(op(2,N),subs(b=a,e=d,new),op(2,M));
      new:=braket(op(3,N),subs(c=a,f=d,new),op(3,M));
   elif nops(N) = 2 then
      new:=braket(op(1,N),v,op(1,M));
      new:=braket(op(2,N),subs(b=a,e=d,new),op(2,M));
   else
       new:=FAIL;
   end if;
elif type(N,list) xor type(M,list) then
   new:=FAIL;
else
   new:=braket(N,v,M);
end if;
new;
end proc;
```

The following code uses the algorithms above to evaluate the second order perturbation correction.

```
bkt:=proc(bra,oper,ket)

##Accepts linear combinations of ##unperturbed states and
evaluates the inner product

local M,N,answer,i,j; M:=op(1,bra[2]); N:=op(1,ket[2]); answer:=0;
for i from 0 to M do
 for j from 0 to N do
 answer:=answer + bra[1][i+1]*ket[1][j+1]*bracket([i,M-i],oper,[j,N-j]);
 end do;
end do; answer; end proc;



##Calculate Matrix Elements
Vmnalpha:=(N,mm,n)->bracket([(N+mm)/2,(N-mm)/2],
         Valpha,[(N+n)/2,(N-n)/2])/alpha/m/h^2/omega^2;
Vmnalpha2:=(N,mm,n)->bracket([(N+mm)/2,(N-mm)/2],
         Valpha2,[(N+n)/2,(N-n)/2])/alpha^2/m^2/h^3/omega^3;

##Find the new basis vectors
for _N from 1 to 8 do
Degalpha:=matrix(_N+1,_N+1):
for n from 0 to _N do
   for mm from 0 to _N do
      Degalpha[mm+1,n+1]:=Vmnalpha(_N,-_N + 2*mm,-_N + 2*n):
   end do:
```

```
end do: _nev:=nops({eigenvalues(Degalpha)}): _EV:=array(1.._nev):
_EVl:=[eigenvectors(Degalpha)]:
for i from 1 to _nev do
   _EV[i]:=_EVl[i]:
end do: for i from 1 to _nev do
   unassign('deg');
   if _EV[i][2] = 2 then
      _V:=convert(_EV[i][3],list);
      _deg[1]:=_V[1];
      _deg[2]:=_V[2];
      Alpha:=infinity;
      beta:=infinity;
      for j from 1 to min((_N+1)/2, _N/2) do
        if _deg[2][j] <> _deg[2][_N+2-j] then
         Alpha:=(_deg[1][_N+2-j] - _deg[1][j])/(_deg[2][j]-_deg[2][_N+2-j]);
         break;
        end if;
      end do;
      for j from 1 to min((_N+1)/2,_N/2) do
        if _deg[2][j] <> -_deg[2][_N+2-j] then
         beta:=(-_deg[1][_N+2-j] - _deg[1][j])/(_deg[2][j]+_deg[2][_N+2-j]);
         break;
        end if;
      end do;
      if Alpha = infinity then
        Even[i]:=normalize(_deg[2]);
```

```
      else
        Even[i]:=normalize(evalm(_deg[1] + Alpha*_deg[2]));
      end if;
      if beta = infinity then
        Odd[i]:=normalize(_deg[2]);
      else
        Odd[i]:=normalize(evalm(_deg[1] + beta*_deg[2]));
      end if;
   elif _EV[i][2] = 1 then
      Even[i]:=normalize(convert(_EV[i][3],list)[1]);
      Odd[i]:=evalm(0*Even[i]);
   end if;
   Even[i][1]:=evalm(Even[i]):
   Odd[i][1]:=evalm(Odd[i]):
   Even[i][2]:=[_N,i,0]:
   Odd[i][2]:=[_N,i,1]:
end do:

##Construct the states that influence the second order correction
i:=1:
if _N < 4 then
 for N from 0 to _N - 1 do
  for n from 1 to N + 1 do
   K[i][2]:=[N];
   for nn from 1 to N + 1 do
    if nn = n then
```

```
     K[i][1][nn]:=1;
    else
     K[i][1][nn]:=0;
    end if;
   end do;
   i:=i+1;
  end do;
 end do;
else
 for N from _N-4 to _N - 1 do
  for n from 1 to N + 1 do
   K[i][2]:=[N];
   for nn from 1 to N + 1 do
    if nn = n then
     K[i][1][nn]:=1;
    else
     K[i][1][nn]:=0;
    end if;
   end do;
   i:=i+1;
  end do;
 end do;
end if;
for N from _N+1 to _N + 4 do
 for n from 1 to N + 1 do
  K[i][2]:=[N];
```

```
  for nn from 1 to N + 1 do
   if nn = n then
    K[i][1][nn]:=1;
   else
    K[i][1][nn]:=0;
   end if;
  end do;
  i:=i+1;
 end do;
end do;
i:=i-1:

##Evaluate the second order perturbation \index{perturbation
theory}

for k from 1 to _nev do
  P1alpha1E[k]:=bkt(Even[k],Valpha,Even[k]):
  P1alpha2E[k]:=bkt(Even[k],Valpha2,Even[k]):
  P2alpha1E[k]:=0:
  for j from 1 to i do
    P2alpha1E[k]:=P2alpha1E[k] +  bkt(K[j],Valpha,Even[k])*
           bkt(Even[k],Valpha,K[j])/h/omega/(_N-K[j][2][1]):
  end do;
  P1alpha1O[k]:=bkt(Odd[k],Valpha,Odd[k]):
  P1alpha2O[k]:=bkt(Odd[k],Valpha2,Odd[k]):
  P2alpha1O[k]:=0:
```

```
  for j from 1 to i do
    P2alpha1O[k]:=P2alpha1O[k] + bkt(K[j],Valpha,Odd[k])*
          bkt(Odd[k],Valpha,K[j])/h/omega/(_N-K[j][2][1]):
  end do:
  print(h*omega*(_N + 1)); print("Even",k,"perturbation \index{perturbation theory}")
  print(simplify(P1alpha1E[k] + P2alpha1E[k] + P1alpha2E[k]));
  print("Odd",k,"perturbation \index{perturbation theory}"); print(simplify(P1alpha1O
  P2alpha1O[k] + P1alpha2O[k]));
end do:
end do:
```

Appendix C

Perturbation Theory

In this appendix, we present some results from perturbation theory in quantum mechanics. The purpose of this presentation is to solidify the notation used in the main body of the thesis and not to present a derivation of the formulas. In this work I follow the conventions of Townsend. [19] An outline of the derivation of these formulas can be found in that work or in any other introductory text on quantum mechanics.

Consider a Hamiltonian, H_0 with eigenstates denoted by $|\phi_n^{(0)}\rangle$ with corresponding eigenvalues given by $E_n^{(0)}$. If a second Hamiltonian, H is related the first by $H = H_0 + \lambda H_1$, where H_1 is some (small) perturbation, then we consider the eigenstates of H, which we denote by $|\psi_n\rangle$ and express them as a power series in the parameter λ:

$$|\psi_n\rangle = |\phi_n^{(0)}\rangle + \lambda|\phi_n^{(1)}\rangle + \lambda^2|\phi_n^{(2)}\rangle + ... \tag{C.1}$$

Similarly, we express the energy eigenvalues of H, denoted by E_n as a power series in λ:

$$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + ... \tag{C.2}$$

These expressions define $|\phi_n^{(i)}\rangle$ and $E_n^{(i)}$. In practice, the parameter λ is used only to keep track of the order of the perturbation and it is set equal to unity after the

calculations.

First order corrections to the energy and to the eigenstates: are

$$E_n^{(1)} = \langle n|\ H_1\ |n\rangle \tag{C.3}$$

$$|\phi_n^{(1)}\rangle = \sum_{k\neq n} |\phi_k^{(0)}\rangle \frac{\langle \phi_k^{(0)}|\ H_1\ |\phi_n^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}}. \tag{C.4}$$

The second order energy correction is given by

$$E_n^{(2)} = \sum_{k\neq n} \frac{\langle \phi_k^{(0)}|\ H_1\ |\phi_n^{(0)}\rangle\ \langle \phi_n^{(0)}|\ H_1\ |\phi_k^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}}. \tag{C.5}$$

Following Townsend's method for calculating the first and second order shift, one could, in principle, calculate any order correction to the energy or eigenstate; however, the formulas are increasingly complex and Townsend does not give them. Third-order energy corrections are occasionally given [26,27]. We found the third and fourth order energy corrections from Nicholas Wheeler, who developed a method for calculating the energy corrections without calculating the corrections to the eigenstates [28]. The third and fourth order energy corrections are given below:

$$E_n^{(3)} = \sum_{i,j\neq n} \frac{V_{ni}V_{ij}V_{jn}}{D_{in}D_{jn}} - V_{nn}\sum_{i\neq n}\frac{V_{ni}V_{in}}{Din^2} \tag{C.6}$$

$$\begin{aligned} E_n^{(4)} = &\left[\sum_{i\neq n}\frac{V_{ni}V_{in}}{D_{in}}\right]\left[\sum_{j\neq n}\frac{V_{nj}V_{jn}}{D_{jn}^2}\right] - V_{nn}^2\sum_{j\neq n}\frac{V_{nj}V_{jn}}{D_{jn}^3} \\ &+V_{nn}\sum_{i,j\neq n}\left[\frac{1}{D_{in}^2 D_{jn}} + \frac{1}{D_{in}D_{jn}^2}\right] V_{ni}VijVjn \\ &-\sum_{i,j,k\neq n}\frac{V_{ni}V_{ij}V_{jk}V_{kn}}{D_{in}D_{jn}D_{kn}}, \end{aligned} \tag{C.7}$$

where we have introduced the shorthand notation $V_{ij} = \langle \phi_i^{(0)}|\ H_1\ |\phi_j^{(0)}\rangle$ and $D_{ij} = E_i^{(0)} - E_j^{(0)}$.

If there are degeneracies of the energy levels of the unperturbed Hamiltonian, then the treatment has to be modified. The problem arises in the first order correction

of the eigenstates, and consequently, the second order correction of the energy. The term in the denominator of the sum $E_n^{(0)} - E_k^{(0)}$ has singularities if the unperturbed state has degenerate energies. Qualitatively, one can understand this to mean that the perturbation series diverges because the basis of eigenstates of the unperturbed Hamiltonian does not converge uniformly to the basis of eigenstates of the perturbed Hamiltonian. To remedy the problem, one needs to find the correct linear combination of degenerate basis states, such that the numerator in the summation $\langle \phi_k^{(0)} |\ H_1\ | \phi_n^{(0)} \rangle$ vanishes whenever the denominator does. This amounts to diagonalizing the matrix whose elements are $\langle \phi_k^{(0)} |\ H_1\ | \phi_n^{(0)} \rangle$. If this diagonalization process removes the degeneracy then one can calculate the perturbation corrections using the new basis as one would in the nondegenerate case.

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