SPIN CURRENT:
THE CONTRIBUTION OF SPIN TO THE PROBABILITY CURRENT FOR NONRELATIVISTIC PARTICLES

by

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For Baby

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Abstract

Calculating the probability current for nonrelativistic particles with spin, using the same procedure as is used for particles without spin, yields an ambiguous result. We resolve this ambiguity without appealing to relativistic quantum mechanics. A unique expression for the probability current of nonrelativistic particles with spin is derived. This expression includes an extra term that arises due to the spin of the particle. We verify that this extra spin current term is not a relativistic effect, and analyze the properties of the spin current for the case of an electron in a homogeneous magnetic field.

Chapter 1

Introduction

Quantum mechanics provides a description of nature almost as surprising and wonderful as nature itself. A critical property, absent in classical mechanics and exposed only through quantum theory, is the quantum spin of an individual particle.[1] Many natural phenomena can only be understood and described in terms of this new property. The characteristic chemical properties of the elements, and their ordering in the periodic table are governed by the spin of the electrons surrounding the nucleus. Spin has explained many observed phenomena including atomic fine structure, magnetism in solids, and superconductivity. Spin physics, when coupled with particle statistics, has predicted surprising new effects such as the recently observed Bose-Einstein condensates.[2] Computer engineers are seeking to use the spin states of individual particles for information storage, heralding the advent of a new generation of supercomputing.[3]

The spin property is arguably the most significant contribution made by quantum theory to our understanding of the physical world. The accurate and consistent description of nature provided by spin, as well as the myriad applications of spin in modern technology, have merited extensive investigations of its properties and operations. Experiments have exposed how the spin property affects the behavior of particles and their interactions, and

[1] For a review on theories of spin in classical mechanics see *Kinematical Theory of Spinning Particles* by Martin Rivas [**?**].

[2] *Physics Today* reports the first observation of Bose-Einstein condensates in an article entitled “Gaseous Bose-Einstein condensate finally observed” [**?**].

[3] See Gershenfeld’s popular article “Quantum computing with molecules,” in *Scientific American* [**?**].

corresponding theory has been promulgated.

The interpretation of the modulus square of the quantum mechanical wave function as the probability density distribution is one of the fundamental tenets of quantum theory. Surprising, then, is the dearth of written material that examines the contribution made by spin to the probability density current. Katsunori Mita states this observation in a recent article to *American Journal of Physics* in this way:

> The spin current is a concept not often treated in textbooks of quantum mechanics, appearing in a very small number of texts. (...) The lack of coverage is also reflected in this journal. We again find only a couple of papers on the spin current.
>
> [**?**]

G. Parker submitted a paper in 1984 to *American Journal of Physics* in which he uses spin current to derive the hyperfine interaction in hydrogen [**?**]. A derivation of the spin current term also appears as an exercise problem in *Schaum's Outlines: Quantum Mechanics* [**?**] but the implications of this term are never discussed nor explored.

A possible reason for this lack of treatment of spin and its effects on probability current, is the peculiar historical development of spin theory that has mislead many into regarding the spin property as a purely relativistic effect. This misconception would belie the idea of investigating the contribution of spin to the probability current of nonrelativistic particles. Furthermore, unexpected difficulties arise when calculating the spin current for nonrelativistic particles with spin.

This thesis seeks to investigate and discuss the effects of spin on the probability current, particularly in relation to the quantum mechanics of nonrelativistic particles. These purposes will be addressed within a format that provides a detailed presentation of all concepts and calculations involved. A thorough background, derivations of all results and calculations, and explanations of relationships between results will be provided. Key derivations are included in the body of the text; while others, less significant to the crux of the argument, are presented in the Appendix for readers to examine at their discretion.

Chapter 2

Statement of the Problem

Introductory quantum mechanics texts seldom treat the quantum mechanical probability current for nonrelativistic particles with spin. The procedure for calculating probability current is most often presented in relation to particles described by the Schrödinger equation, that is particles *without* spin. In later chapters, after having introduced readers to the Pauli equation and its description of particles *with* spin, the authors rarely return to the topic of probability current.[1] This omission may cause readers to incorrectly infer that the calculation of the probability current for particles with spin follows the same line of argument as is used for particles without spin.

A closer inspection of the derivation of the probability current for particles with spin reveals that adopting the same procedure as is used for particles without spin gives rise to a nontrivial ambiguity. Indeed, we discover that applying the standard procedure to the Pauli equation fails to determine the probability current uniquely. We may add an extra term to the resultant expression for the probability current with impunity, and still satisfy the continuity equation. This result is intolerable since the probability current is physically measurable and must be uniquely defined. We must conclude that the standard approach used for calculating the probability current of the Schrödinger equation is incapable of deriving the unique probability current for particles with spin.

In a recent paper submitted to *American Journal of Physics*, Marek Nowakowski addresses this ambiguity. Nowakowski argues that a nonrelativistic reduction of the probability current for relativistic particles with spin

[1] See for example a text by D. Griffiths titled *Introduction to Quantum Mechanics* [**?**].

reveals the correct and unique expression for the probability current of nonrelativistic particles with spin. His calculations show that the spin of the particle does indeed introduce an extra term to the probability current of the form $\frac{\hbar}{2m}\nabla \times (\psi^{\dagger}\sigma\psi)$. This extra term is called the *spin current*: the contribution to the probability current that arises due to the spin of the particle.

Nowakowski's method for deriving the extra spin current term requires the use of relativistic quantum mechanics. Indeed, he regards this approach as imperative in order to resolve the ambiguity:

> This ambiguity cannot be resolved by means of nonrelativistic quantum mechanics alone. Or, to put it differently, this ambiguity only appears from the point of view of nonrelativistic quantum mechanics. (...) Hence, there is *a priori* no way to decide from the point of view of nonrelativistic quantum mechanics whether a term should be added to the [probability current] or not.
>
> [**?**]

The purpose of the present research is to derive the unique probability current, including the spin current term, for nonrelativistic particles with spin without appealing to a relativistic theory. Such a derivation will highlight the fact that spin current is not a relativistic effect. Having established the nonrelativistic nature of spin current, the research will investigate the properties and behavior of spin current. Specifically, we shall investigate the contributions of the extra spin current term for the case of an electron in a homogeneous magnetic field.

Chapter 3

Background

3.1 Quantum Mechanical Wave Equations

Quantum mechanics attributes to matter the properties of waves. The postulate that matter behaves as a wave and can be described by a wave function, provides a mathematical description or model of many previously unexplained physical phenomena such as the diffraction pattern of an electron beam. Although the wave function itself bears no physical significance, information about all physically observable quantities is contained within the wave function. The wave functions are solutions to quantum mechanical wave equations.

The present research is sensitive to the different properties of the various wave equations. For this reason, a brief description of the wave equations is presented here.

3.1.1 The Schrödinger equation

The Schrödinger equation, a nonrelativistic second-order differential equation, is the simplest of the quantum mechanical wave equations:

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi. \tag{3.1}$$

When the Schrödinger equation is solved for a specific potential and boundary conditions are included, the solutions reveal that the energy is quantized in discrete values, or energy levels. Zeeman noticed that in the presence of a magnetic field each of the energy levels for the electrons in hydrogen splits into two levels. The property that causes this degeneracy in the energy levels is termed *spin*. The Schrödinger equation fails to predict the splitting of energy levels in hydrogen since spin is absent in this equation.

Spin is an intrinsic property of the all particles that contributes to its total angular momentum in such a way as to ensure conservation of the angular momentum [**?**]. Spin is a mathematical construct that yields the correct predictions from the equations when comparing with observations in nature. Texts on quantum mechanics typically do not provide a physical interpretation of what spin actually is, or what it looks like. It is simply a mathematical property ascribed to a particle in order to find agreement with experimental results.[1] Spin is nevertheless a powerful predictor of physical phenomena. The fact that spin cannot be dismissed simply due to lack of physical interpretation is evidenced by its measurable and tangible influence on experiments such as the Stern-Gerlach experiment. In this experiment, a beam of silver atoms passing through an inhomogeneous magnetic field, splits into two distinct beams corresponding to quantized spin states of spin up and spin down, contrary to the smooth distribution predicted by classical theory.

3.1.2 The Pauli equation

Pauli found that he could recover the data of the Zeeman experiment if he appended an extra spin term of the form $\frac{e\hbar}{2m}\sigma.\mathbf{B}$ to the Schrödinger equation:

$$i\hbar\partial_t\psi = \left[-\frac{\hbar^2}{2m}\nabla^2 - \frac{e\hbar}{2m}\sigma.\mathbf{B}\right]\psi. \tag{3.2}$$

This extra term introduced non-commuting objects, σ, or spin operators that act on a space of two-component wave functions called spinors. The non-commutative property of the spin operators requires that the spinors be at least two-dimensional. This two-component nature of the wave functions allows for two solutions of differing energies. Thus, Pauli's amendment to the Schrödinger equation yields two solutions of different energies corresponding to the two different spin states of spin up, or spin down. The two energy solutions agree with the observation by Zeeman that each energy level is split into two.

[1] Ohanian proposes a physical interpretation of spin in his paper "What is spin" [**?**].

The addition of the extra term in Pauli's equation provides the desired mathematical result, yet appears as a forceful addition of the spin property by a hand-placed constraint. A more appealing derivation of the spin property followed later with Dirac's relativistic approach.

3.1.3 The Dirac equation

Dirac incorporated the principles of relativity into quantum mechanics by deriving a relativistic wave equation that was first-order both in time and space.[2] This approach intrinsically requires the non-commuting properties of the spin operators. Hence, Dirac was able to derive spin from information contained within the fundamental wave equation rather than hand-placing it in the equation *a la* Pauli.[3] The Dirac equation is a relativistic wave equation for particles with spin:

$$\left[(\alpha.\mathbf{p})c + \beta mc^2\right]\Psi = E\Psi. \tag{3.3}$$

In agreement with the correspondence principle, a nonrelativistic reduction of the Dirac equation leads to the Pauli equation.[4]

Table **??** classifies these wave functions according to their properties. Mentioned in the table, but not pertinent to our study, is the Klein-Gordon equation, a second-order wave equation describing relativistic particles without spin.

3.2 Probability Current

Probability theory, although inherently reluctant to reveal exactitudes of outcomes, provides a powerful language for modeling nature. The amplitude of the modulus square of the wave function is interpreted as a probability density distribution. Regions of largest amplitude represent locations where the particle is most likely to be found. The sum of the probabilities over all space should be unity, corresponding to the fact the probability of finding the particle somewhere in all space must be one. The time dependence of the wave equation governs the propagation of the wave. Regions of largest probability

[2] See Appendix **??** Derivation of the Dirac equation.
[3] See Appendix **??** Derivation of spin from the Dirac equation.
[4] See Appendix **??** Nonrelativistic limits of the Dirac equation.

Table 3.1: Properties of the Schrödinger, Pauli, and Dirac wave equations.

density will move through space as determined by the wave equation. The movement or flow of probability density is termed *probability current.*

The standard procedure for deriving an expression for the probability current is most easily demonstrated using the Schrödinger equation. We remember that the wave function, ψ, of a particle contains all information about physical observables, (including the probability current) and that the square of the wave function is interpreted as the probability density,

$$\rho \equiv \psi^* \psi, \tag{3.4}$$

where ψ^* represents the complex conjugate of ψ.

Calculating the probability current (represented by the letter $\mathbf{J}$) requires that we manipulate the Schrödinger equation,

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi, \tag{3.5}$$

into a form in which we can compare it with the continuity equation,

$$\partial_t \rho + \nabla . \mathbf{J} = 0. \tag{3.6}$$

The continuity equation is simply a statement of the conservation of probability density. A local change in probability density arises only due to an inward or outward flow of probability current. Probability for finding the particle cannot be created nor destroyed. Once the Schrödinger equation is in a form comparable to the continuity equation, we can conveniently read off the probability current, **J**.

3.2.1 Probability current for the Schrödinger equation

We begin with the definition of probability density,

$$\rho \equiv \psi^* \psi, \tag{3.7}$$

and differentiate with respect to time,

$$\partial_t \rho = \psi^* \partial_t \psi + \psi \partial_t \psi^*. \tag{3.8}$$

Now we write the Schrödinger equation,

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi. \tag{3.9}$$

Our program here is to manipulate the Schrödinger equation into a form in which we can compare it with the continuity equation. We proceed by multiplying both sides of the equation by ψ^*:

$$i\hbar \psi^* \partial_t \psi = -\frac{\hbar^2}{2m} \psi^* \nabla^2 \psi + V\psi^* \psi. \tag{3.10}$$

Now we take the complex conjugate of equation (**??**),

$$-i\hbar \psi \partial_t \psi^* = -\frac{\hbar^2}{2m} \psi \nabla^2 \psi^* + V\psi\psi^*. \tag{3.11}$$

Subtracting equation (**??**) from equation (**??**),

$$i\hbar(\psi^* \partial_t \psi + \psi \partial_t \psi^*) = -\frac{\hbar^2}{2m} (\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*), \tag{3.12}$$

and dividing through by $i\hbar$ we find

$$\psi^* \partial_t \psi + \psi \partial_t \psi^* = \frac{i\hbar}{2m} (\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*). \tag{3.13}$$

We have fortuitously manipulated the equation so that the left-hand side is the exact time-derivative of the probability density. We can substitute this definition into the left-hand side. Next, we must work with right-hand side so that we may compare with the continuity equation to read off the expression for **J**.

$$\partial_t \rho = \frac{i\hbar}{2m}(\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*) \tag{3.14}$$

We can add and subtract convenient terms to the right-hand side to obtain

$$\partial_t \rho = \nabla \frac{i\hbar}{2m}(\psi^* \nabla \psi - \psi \nabla \psi^*), \tag{3.15}$$

and then bring both terms to the left-hand side

$$\partial_t \rho + \nabla \left[\frac{i\hbar}{2m}(\psi \nabla \psi^* - \psi^* \nabla \psi) \right] = 0. \tag{3.16}$$

We have arrived at a form that we can compare with the continuity equation,

$$\partial_t \rho + \nabla . \mathbf{J} = 0. \tag{3.17}$$

Apparently, the expression for **J** must be

$$\boxed{\mathbf{J}_{\text{Schrödinger}} = \frac{i\hbar}{2m}(\psi \nabla \psi^* - \psi^* \nabla \psi).} \tag{3.18}$$

We shall now attempt to employ this same procedure to calculate the probability current for nonrelativistic particles with spin. We must remember that our wave functions are solutions to the Pauli equation. These solutions are not scalar functions, but rather two-component spinors. We begin by naively adopting the identical approach for calculating the probability current as we used in the Schrödinger case.

3.2.2 Probability current for the Pauli equation

First we write the Pauli equation:

$$\left[-\frac{\hbar^2}{2m}\nabla^2-\frac{e\hbar}{2m}\sigma.\mathbf{B}\right]\psi=i\hbar\partial_t\psi. \tag{3.19}$$

Just as for the Schrödinger equation, we multiply both sides by $\psi^\dagger$ to get

$$\psi^\dagger\left[-\frac{\hbar^2}{2m}\nabla^2-\frac{e\hbar}{2m}\sigma.\mathbf{B}\right]\psi=i\hbar\psi^\dagger\partial_t\psi, \tag{3.20}$$

$$-\frac{\hbar^2}{2m}\psi^\dagger\nabla^2\psi-\frac{e\hbar}{2m}\psi^\dagger(\sigma.\mathbf{B})\psi=i\hbar\psi^\dagger\partial_t\psi, \tag{3.21}$$

where, because ψ is a two-component wave function,

$$\psi^\dagger=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}^\dagger=\begin{pmatrix}\psi_1^* & \psi_2^*\end{pmatrix}. \tag{3.22}$$

Now we take the complex conjugate of equation (**??**), noting that for three arbitrary operators **A**, **B**, and **C**,

$$(\mathbf{ABC})^\dagger=\mathbf{C}^\dagger\mathbf{B}^\dagger\mathbf{A}^\dagger, \tag{3.23}$$

and that since the spin operators are Hermitian,

$$\sigma^\dagger=\sigma. \tag{3.24}$$

$$-\frac{\hbar^2}{2m}(\nabla^2\psi^\dagger)\psi-\frac{e\hbar}{2m}\psi^\dagger(\sigma.\mathbf{B})\psi=-i\hbar(\partial_t\psi^\dagger)\psi \tag{3.25}$$

Subtracting equation (**??**) from equation (**??**) leaves

$$\frac{\hbar^2}{2m}\left[(\nabla^2\psi^\dagger)\psi-\psi^\dagger(\nabla^2\psi)\right]=i\hbar\left[\psi^\dagger\partial_t\psi+(\partial_t\psi^\dagger)\psi\right]. \tag{3.26}$$

Now we divide through by $i\hbar$,

$$-\frac{i\hbar}{2m}\left[(\nabla^2\psi^\dagger)\psi-\psi^\dagger(\nabla^2\psi)\right]=\psi^\dagger\partial_t\psi+(\partial_t\psi^\dagger)\psi, \tag{3.27}$$

but remember that

$$\partial_t \rho = \psi^\dagger \partial_t \psi + (\partial_t \psi^\dagger)\psi, \tag{3.28}$$

so we have

$$-\frac{i\hbar}{2m}\Big[(\nabla^2\psi^\dagger)\psi - \psi^\dagger(\nabla^2\psi)\Big] = \partial_t \rho. \tag{3.29}$$

Again, we have conveniently manipulated the right-hand side to be the exact time derivative of the probability density. Minor rearrangement of terms will allow us to compare with the continuity equation.

$$\partial_t \rho + \nabla . \frac{i\hbar}{2m}\Big[(\nabla\psi^\dagger)\psi - \psi^\dagger(\nabla\psi)\Big] = 0 \tag{3.30}$$

Comparing this with the continuity equation,

$$\partial_t \rho + \nabla . \mathbf{J} = 0, \tag{3.31}$$

suggests the probability current to be

$$\boxed{\mathbf{J}_{\text{Pauli}} = \frac{i\hbar}{2m}\Big[(\nabla\psi^\dagger)\psi - \psi^\dagger(\nabla\psi)\Big].} \tag{3.32}$$

It would appear that we have arrived at a valid and conclusive expression for the probability current of the Pauli equation. However, a careful inspection of the result will uncover the ambiguity. Because the divergence of a curl is zero, the continuity equation in line (**??**) is still satisfied if we append any term to the probability current of the form $\nabla \times \mathbf{v}$, where $\mathbf{v}$ is any vector or multi-component object. The Pauli equation acts on a space that includes such objects, namely σ, and so would, in principle, allow for the construction of an extra curl term of the form $\nabla \times (\psi^\dagger \sigma \psi)$.

$$\mathbf{J}_{\text{Pauli}} = \frac{i\hbar}{2m}\Big[(\nabla\psi^\dagger)\psi - \psi^\dagger(\nabla\psi)\Big] \quad + \quad (?)\nabla \times (\psi^\dagger \sigma \psi) \tag{3.33}$$

The constraints of the continuity equation are insufficient to uniquely determine the expression for the probability current. This ambiguity is a moot point in the derivation of the Schrödinger current since the Schrödinger equation involves only scalar functions with which the creation of an extra curl term is impossible. The standard approach for calculating the probability current is clearly inadequate in calculating the unique probability current for nonrelativistic particles with spin.

Nowakowski argues that the resolution of the ambiguity must be found using relativistic quantum mechanics. He proposes that a nonrelativistic reduction of the Dirac probability current (the probability current for *relativistic* particles with spin) will disclose the correct form of the extra term to be appended to the probability current for *nonrelativistic* particles with spin. Introducing the Lorentz symmetry of relativity imposes additional constraints that "restrict the number of possible terms and can therefore resolve an otherwise persistent ambiguity" [**?**].

In order to follow Nowakowski's argument we begin with the probability current for the Dirac equation,[5]

$$\mathbf{J}_{\text{Dirac}} = c(\Psi^{\dagger}\alpha\Psi). \tag{3.34}$$

Now we must take the nonrelativistic limit of this current to find the correct probability current for nonrelativistic particles with spin.

3.2.3 Nonrelativistic limit of the Dirac current

We begin with the Dirac current,

$$\mathbf{J}_{\text{Dirac}} = c(\Psi^{\dagger}\alpha\Psi), \tag{3.35}$$

where Ψ is a four-element column matrix consisting of two two-component spinors ψ and χ,

$$\Psi \equiv \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \tag{3.36}$$

and α is a set of three 4×4 matrices for which we choose the standard representation,

$$\alpha_k = \begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix}. \tag{3.37}$$

Placing these terms explicitly in the expression for the Dirac current and performing the matrix multiplication yields

[5]See Appendix **??** Derivation of the probability current for the Dirac equation.

$$J_{\text{Dirac}_k} = c[\begin{pmatrix} \psi^\dagger & \chi^\dagger \end{pmatrix} \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix}] \tag{3.38}$$

$$J_{\text{Dirac}_k} = c[\begin{pmatrix} \psi^\dagger & \chi^\dagger \end{pmatrix} \begin{pmatrix} \sigma_k\chi \\ \sigma_k\psi \end{pmatrix} \tag{3.39}$$

$$J_{\text{Dirac}_k} = c[\psi^\dagger\sigma_k\chi + \chi^\dagger\sigma_k\psi]. \tag{3.40}$$

Equation (**??**) is still a relativistic expression for the Dirac current. We must examine the Dirac equation to see how we shall reduce this expression to a nonrelativistic limit.

The Dirac equation reads

$$\left[(\alpha.\mathbf{p})c + \beta mc^2\right]\Psi = E\Psi. \tag{3.41}$$

The nonrelativistic limit will be easier to assess if shift the origin of energy to mc^2. We may do this by defining

$$\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \begin{pmatrix} e^{-\frac{imc^2}{\hbar}}\tilde{\psi} \\ e^{-\frac{imc^2}{\hbar}}\tilde{\chi} \end{pmatrix} \tag{3.42}$$

where each of the components includes a phase factor that explicitly removes the rest mass, the difference between nonrelativistic and relativistic energies.

We place this wave function into the Dirac equation and perform the matrix multiplication,

$$\begin{aligned} &\begin{bmatrix} 0 & (\sigma.\mathbf{p})c \\ (\sigma.\mathbf{p})c & 0 \end{bmatrix} \begin{pmatrix} e^{-\frac{imc^2}{\hbar}}\tilde{\psi} \\ e^{-\frac{imc^2}{\hbar}}\tilde{\chi} \end{pmatrix} \\ +&\begin{bmatrix} mc^2 & 0 \\ 0 & -mc^2 \end{bmatrix} \begin{pmatrix} e^{-\frac{imc^2}{\hbar}}\tilde{\psi} \\ e^{-\frac{imc^2}{\hbar}}\tilde{\chi} \end{pmatrix} = E \begin{pmatrix} e^{-\frac{imc^2}{\hbar}}\tilde{\psi} \\ e^{-\frac{imc^2}{\hbar}}\tilde{\chi} \end{pmatrix} \end{aligned} \tag{3.43}$$

$$\begin{aligned} c(\sigma.\mathbf{p})e^{-\frac{imc^2}{\hbar}}\tilde{\chi} + mc^2 e^{-\frac{imc^2}{\hbar}}\tilde{\psi} &= Ee^{-\frac{imc^2}{\hbar}}\tilde{\psi} \\ c(\sigma.\mathbf{p})e^{-\frac{imc^2}{\hbar}}\tilde{\psi} - mc^2 e^{-\frac{imc^2}{\hbar}}\tilde{\chi} &= Ee^{-\frac{imc^2}{\hbar}}\tilde{\chi}. \end{aligned} \tag{3.44}$$

Now we substitute the time-derivative operator for E,

$$c(\sigma.\mathbf{p})e^{-\frac{imc^2}{\hbar}t}\tilde{\chi} + mc^2 e^{-\frac{imc^2}{\hbar}t}\tilde{\psi} = i\hbar\partial_t\left(e^{-\frac{imc^2}{\hbar}t}\tilde{\psi}\right) \tag{3.45}$$
$$c(\sigma.\mathbf{p})e^{-\frac{imc^2}{\hbar}t}\tilde{\psi} - mc^2 e^{-\frac{imc^2}{\hbar}t}\tilde{\chi} = i\hbar\partial_t\left(e^{-\frac{imc^2}{\hbar}t}\tilde{\chi}\right),$$

and evaluate the time-derivative,

$$\left[c(\sigma.\mathbf{p})\tilde{\chi} + mc^2\tilde{\psi}\right]e^{-\frac{imc^2}{\hbar}t} = i\hbar\left[-\frac{imc^2}{\hbar}\tilde{\psi} + (\partial_t\tilde{\psi})\right]e^{-\frac{imc^2}{\hbar}t} \tag{3.46}$$
$$\left[c(\sigma.\mathbf{p})\tilde{\psi} - mc^2\tilde{\chi}\right]e^{-\frac{imc^2}{\hbar}t} = i\hbar\left[-\frac{imc^2}{\hbar}\tilde{\chi} + (\partial_t\tilde{\chi})\right]e^{-\frac{imc^2}{\hbar}t}$$

Now we can divide both lines by $e^{-\frac{imc^2}{\hbar}t}$ to get

$$c(\sigma.\mathbf{p})\tilde{\chi} + mc^2\tilde{\psi} = mc^2\tilde{\psi} + i\hbar\partial_t\tilde{\psi} \tag{3.47}$$
$$c(\sigma.\mathbf{p})\tilde{\psi} - mc^2\tilde{\chi} = mc^2\tilde{\chi} + i\hbar\partial_t\tilde{\chi}$$

$$c(\sigma.\mathbf{p})\tilde{\chi} = i\hbar\partial_t\tilde{\psi} \tag{3.48}$$
$$c(\sigma.\mathbf{p})\tilde{\psi} - 2mc^2\tilde{\chi} = i\hbar\partial_t\tilde{\chi}.$$

In the nonrelativistic limit we assume that the kinetic energy (the right-hand side of the equation) is small compared to the mass. In this limit, we read off the bottom equation in line (**??**) as

$$c(\sigma.\mathbf{p})\tilde{\psi} - 2mc^2\tilde{\chi} \approx 0. \tag{3.49}$$

Inserting the gradient operator for $\mathbf{p}$, we may rewrite this as

$$\tilde{\chi} \approx \frac{1}{2mc}\sigma.(\frac{\hbar}{i}\nabla)\tilde{\psi} \tag{3.50}$$

$$\tilde{\chi} \approx -\frac{i\hbar}{2mc}\sigma.\nabla\tilde{\psi}. \tag{3.51}$$

This is our expression for $\tilde{\chi}$ in the nonrelativistic limit. We will substitute this into the expression for the probability current in line (**??**). Notice that since the probability current contains both ψ and $\psi^\dagger$ as well as χ and $\chi^\dagger$ it is insensitive to the phase of the wave functions. Thus, we drop the tildes on $\tilde{\psi}$ and $\tilde{\chi}$ and revert to our original wave functions ψ and χ.

$$\mathrm{J}_{\mathrm{DiracNR}_k} = c\left[\psi^\dagger\sigma_k\left[-\frac{i\hbar}{2mc}\sigma.\nabla\psi\right] + \left[-\frac{i\hbar}{2mc}\sigma.\nabla\psi\right]^\dagger\sigma_k\psi\right]. \tag{3.52}$$

We can expand out the terms to find

$$\mathrm{J}_{\mathrm{DiracNR}_k} = \frac{i\hbar}{2m}(\nabla_i\psi^\dagger)\sigma_i\sigma_k\psi - \frac{i\hbar}{2m}\psi^\dagger\sigma_k\sigma_i\nabla_i\psi. \tag{3.53}$$

Now we refer to the properties of the σ matrices given in Appendix **??**,

$$\begin{aligned}\sigma_i\sigma_k &= \delta_{ik} + i\epsilon_{ikl}\sigma_l \\ \sigma_k\sigma_i &= \delta_{ki} - i\epsilon_{ikl}\sigma_l,\end{aligned} \tag{3.54}$$

so that our equation becomes

$$\mathrm{J}_{\mathrm{DiracNR}_k} = \frac{i\hbar}{2m}(\nabla_i\psi^\dagger)(\delta_{ik} + i\epsilon_{ikl}\sigma_l)\psi - \frac{i\hbar}{2m}\psi^\dagger(\delta_{ki} - i\epsilon_{ikl}\sigma_l)\nabla_i\psi. \tag{3.55}$$

Expanding out each term,

$$\mathrm{J}_{\mathrm{DiracNR}_k} = \frac{i\hbar}{2m}(\nabla_k\psi^\dagger)\psi - \frac{\hbar}{2m}\psi^\dagger\epsilon_{ikl}\sigma_l\nabla_i\psi - \frac{i\hbar}{2m}\psi^\dagger\nabla_k\psi - \frac{\hbar}{2m}(\nabla_i\psi^\dagger)\epsilon_{ikl}\sigma_l\psi, \tag{3.56}$$

and now gathering like terms and simplifying yields

$$\mathrm{J}_{\mathrm{DiracNR}_k} = \frac{i\hbar}{2m}\left[(\nabla_k\psi^\dagger)\psi - \psi^\dagger\nabla_k\psi\right] - \frac{\hbar}{2m}(\psi^\dagger\epsilon_{ikl}\sigma_l\nabla_i\psi + (\nabla_i\psi^\dagger)\epsilon_{ikl}\sigma_l\psi) \tag{3.57}$$

$$\mathrm{J}_{\mathrm{DiracNR}_k} = \frac{i\hbar}{2m}\left[(\nabla_k\psi^\dagger)\psi - \psi^\dagger\nabla_k\psi\right] + \frac{\hbar}{2m}(\psi^\dagger\epsilon_{ilk}\sigma_l\nabla_i\psi + (\nabla_i\psi^\dagger)\epsilon_{ilk}\sigma_l\psi) \tag{3.58}$$

$$\mathrm{J}_{\mathrm{DiracNR}_k} = \frac{i\hbar}{2m}\left[(\nabla_k\psi^\dagger)\psi - \psi^\dagger\nabla_k\psi\right] + \frac{\hbar}{2m}\left(\psi^\dagger[\nabla\times\sigma\psi]_k + [\nabla\times\psi^\dagger\sigma]_k\psi\right). \tag{3.59}$$

These two terms form the final result for our nonrelativistic limit of the Dirac current,

$$\boxed{\mathbf{J}_{\mathrm{DiracNR}} = \frac{i\hbar}{2m}\left[(\nabla\psi^\dagger)\psi - \psi^\dagger\nabla\psi\right] + \frac{\hbar}{2m}\left[\nabla\times(\psi^\dagger\sigma\psi)\right].} \tag{3.60}$$

$$\mathbf{J}_{\mathrm{DiracNR}} = \mathbf{J}_{\mathrm{Pauli}} + \mathbf{J}_{\mathrm{spin}} \tag{3.61}$$

When we compare our final result here with the result for the Pauli probability current (**??**), the extra spin current term is easily apparent. Evidently, an extra term containing a curl of σ must indeed be added to the Pauli current. Furthermore, the Dirac equation predicts the unique coefficient of the extra spin current term to be $\frac{\hbar}{2m}$. As indicated by Nowakowski, the introduction of the Lorentz symmetry provides a unique and unambiguous expression for the probability current of nonrelativistic particles with spin.

The derivations of the probability currents for each of the three equations discussed reveals that the probability currents do not obey the same limiting operations as the equations themselves. A nonrelativistic limit of the Dirac equation yields the Pauli equation; however, a nonrelativistic limit of the Dirac probability current yields an extra term when compared with the Pauli probability current. Figure **??** on page **??** illustrates this inconsistency between the equations and their currents. The calculation of the probability current using the Dirac equation depends on whether we take the nonrelativistic limit before or after calculating the probability current.

The fact that spin contributes to the probability current of nonrelativistic particles is a significant finding that has not been widely documented. However, this result alone is not the sole objective of this paper. In addition to an investigation of this spin current term, we are also concerned with the method in which it is derived.

Nowakowski's relativistic prescription for resolving the ambiguity implies that the nonrelativistic theory does not contain sufficient information to provide the necessary constraints. The reader may be left to wonder if this is

Figure 3.1: The probability current for nonrelativistic particles with spin can be derived from the Dirac equation by calculating the respective current and performing a nonrelativistic reduction. However, as this diagram illustrates, the result depends on the order in which these two procedures is performed. Making the nonrelativistic reduction after calculating the Dirac current introduces an extra term, $\frac{\hbar}{2m}\nabla \times (\psi^{\dagger}\sigma\psi)$, when compared with the Pauli current.

because spin current is an inherently relativistic effect not fully describable by nonrelativistic quantum mechanics. On the other hand, if spin current is not at all relativistic, why must we appeal to a relativistic theory to determine its properties? Herein lies the problem and motivation for the ensuing research.

Chapter 4

Research Objectives

Having established a historical background and motivation for this study, we can now make a clear statement of the research objectives:

1. We shall derive the correct probability current, including the spin current term, for nonrelativistic particles with spin without appealing to relativistic theory. This derivation will verify that spin current is not a relativistic effect, and merits further scrutiny in the context of nonrelativistic quantum mechanics.

2. We shall investigate the properties and behavior of spin current by analyzing the contribution of the spin current to the total probability current for an electron in a homogeneous magnetic field.

Chapter 5

Results

5.1 A Nonrelativistic Derivation of Spin Current

Dirac's serendipitous derivation of spin within a relativistic context has led many to believe that spin is a purely relativistic effect. From a historical perspective such a conclusion is apparently justified since spin was absent in the nonrelativistic Schrödinger equation, forcefully introduced in the Pauli equation, and derived naturally only after Dirac's inclusion of relativistic theory. In 1969, however, Jean-Marc Levy-Leblond showed that the spin property can be derived without appealing to special relativity, but by assuming only Galilean invariance [**?**].

The Levy-Leblond equation is obtained by factorizing the Schrödinger equation into two first-order differential equations:

$$\begin{aligned} E\phi - c(\sigma.\mathbf{p})\chi &= 0 \\ 2mc^2\chi - c(\sigma.\mathbf{p})\phi &= 0. \end{aligned} \tag{5.1}$$

This linearization process requires the non-commutative properties of the σ spin matrices. Spin appears as a natural consequence from the constraints present in the equation. Derivation of the Levy-Leblond equation demonstrates emphatically that spin is not a relativistic effect.[1]

[1] See Appendix **??** Derivation of the Levy-Leblond equation.

Table 5.1: Properties of the Levy-Leblond equation in comparison with the Schrödinger, Pauli, and Dirac wave equations.

The Levy-Leblond equation describes the wave functions of nonrelativistic particles with spin. It differs, however, from the Pauli equation in that spin is an intrinsic property of the equation and not a hand-placed constraint.

The Levy-Leblond equation exhibits relativistic correspondence since it can be derived through a nonrelativistic reduction of the Dirac equation.[2] The relationship between the Levy-Leblond equation and the other quantum mechanical wave equations discussed previously is summarized in Table **??**.

The misconception that the spin property arises only in relativistic quantum mechanics may lead to the equally incorrect assumption that spin current is also a relativistic effect. Indeed, Nowakowski's insistence on the need for a relativistic resolution of the ambiguity regarding spin current strengthens this notion. However, Levy-Leblond's derivation of spin without employing relativity theory, hints at the possibility of deriving the correct spin current term utilizing only nonrelativistic quantum mechanics. Surely, if indeed spin

[2] See Appendix **??** Nonrelativistic limit of the Dirac equation.

is not a relativistic effect, then spin current (the contribution to the probability current arising from spin) must be derivable in a purely nonrelativistic context.

We have demonstrated in Section **??** that applying the standard program for calculating the probability current to the Pauli equation fails to generate the extra spin current term. Evidently, the procedure for calculating the probability current fails to preserve the information about spin contained in Pauli's *ad hoc* term. In the Levy-Leblond equation, however, spin is naturally embedded as an intrinsic constraint. If the procedure for calculating the probability current is able to preserve this information about spin contained in the Levy-Leblond equation, and describe the subsequent contributions of spin to the probability current, it will provide a nonrelativistic derivation of spin current. We attempt at once to calculate the probability current for the Levy-Leblond equation using the standard procedure. Our purpose is to observe if the spin current term is produced, and whether it is uniquely defined.

5.1.1 Probability current for the Levy-Leblond equation

We shall follow the same procedure as has been used in previous calculations of the probability current. Each step of the derivation is included in detail to allow a clear analysis of the interactions between the terms, specifically the terms that describe the effects of spin.

As per usual we define the probability density,

$$\rho \equiv \phi^\dagger \phi, \tag{5.2}$$

so that

$$\partial_t \rho = \phi^\dagger (\partial_t \phi) + (\partial_t \phi^\dagger)\phi. \tag{5.3}$$

We will need this to solve for $\mathbf{J}_{\text{Levy-Leblond}}$ in the continuity equation. Next, we write the Levy-Leblond wave equation,

$$\begin{aligned} E\phi - c(\sigma.\mathbf{p})\chi &= 0 \\ 2mc^2\chi - c(\sigma.\mathbf{p})\phi &= 0. \end{aligned} \tag{5.4}$$

We can substitute operators for E and $\mathbf{p}$ to find

$$\begin{aligned} i\hbar\partial_t\phi + i\hbar c(\sigma.\nabla)\chi &= 0 \\ 2mc^2\chi + i\hbar c(\sigma.\nabla)\phi &= 0. \end{aligned} \tag{5.5}$$

From the top line of this wave equation we see that

$$\partial_t\phi = -c(\sigma.\nabla)\chi. \tag{5.6}$$

We can use equation (**??**) and its adjoint,

$$\partial_t\phi^\dagger = -c(\nabla\chi^\dagger.\sigma), \tag{5.7}$$

to substitute for $\partial_t\phi$ and $\partial_t\phi^\dagger$ in our expression for $\partial_t\rho$ in equation (**??**),

$$\partial_t\rho = -c\phi^\dagger(\sigma.\nabla\chi) - c(\nabla\chi^\dagger.\sigma)\phi. \tag{5.8}$$

Furthermore, from the bottom line of the Levy-Leblond equation we see that

$$\chi = -\frac{i\hbar}{2mc}(\sigma.\nabla\phi), \tag{5.9}$$

and hence

$$\chi^\dagger = \frac{i\hbar}{2mc}(\nabla\phi^\dagger).\sigma. \tag{5.10}$$

These expressions for χ and $\chi^\dagger$ can also be substituted into our expression for $\partial_t\rho$,

$$\partial_t\rho = -c\phi^\dagger(\sigma.\nabla\chi) - c(\nabla\chi^\dagger.\sigma)\phi \tag{5.11}$$

$$\partial_t\rho = \frac{i\hbar}{2m}\phi^\dagger\,(\sigma.\nabla)\,(\sigma.\nabla\phi) - \frac{i\hbar}{2m}\left[\nabla(\nabla\phi^\dagger.\sigma).\sigma\right]\phi \tag{5.12}$$

$$\partial_t\rho = \frac{i\hbar}{2m}\left[\phi^\dagger\sigma_i\sigma_j\nabla_i(\nabla_j\phi) - \left(\nabla_i(\nabla_j\phi^\dagger\sigma_j)\sigma_i\right)\phi\right]. \tag{5.13}$$

Now we may add and subtract a convenient term,

$$\partial_t \rho = \frac{i\hbar}{2m}\Big[(\nabla_i \phi^\dagger)\sigma_i\sigma_j(\nabla_j\phi) + \phi^\dagger\sigma_i\sigma_j\nabla_i(\nabla_j\phi) \\ -\nabla_i(\nabla_j\phi^\dagger)\sigma_j\sigma_i\phi - (\nabla_j\phi^\dagger)\sigma_j\sigma_i(\nabla_i\phi)\Big]. \tag{5.14}$$

Using the product rule for derivatives, we can rewrite this as

$$\partial_t \rho = \frac{i\hbar}{2m}\nabla_i\left[\phi^\dagger\sigma_i\sigma_j(\nabla_j\phi) - (\nabla_j\phi^\dagger)\sigma_j\sigma_i\phi\right] \tag{5.15}$$

$$\partial_t \rho + \nabla_i\frac{i\hbar}{2m}\left[(\nabla_j\phi^\dagger)\sigma_j\sigma_i\phi - \phi^\dagger\sigma_i\sigma_j(\nabla_j\phi)\right] = 0. \tag{5.16}$$

Comparing this with the continuity equation,

$$\partial_t \rho + \nabla.\mathbf{J} = 0, \tag{5.17}$$

we can easily determine $\mathbf{J}_{\mathrm{Levy-Leblond}}$,

$$\mathrm{J}_{\mathrm{Levy-Leblond}_i} = \frac{i\hbar}{2m}[(\nabla_j\phi^\dagger)\sigma_j\sigma_i\phi - \phi^\dagger\sigma_i\sigma_j(\nabla_j\phi)]. \tag{5.18}$$

But,

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \tag{5.19}$$

$$\sigma_j\sigma_i = \delta_{ij} - i\epsilon_{ijk}\sigma_k \tag{5.20}$$

So,

$$\mathrm{J}_{\mathrm{Levy-Leblond}_i} = \frac{i\hbar}{2m}\Big[(\nabla_j\phi^\dagger)(\delta_{ij} - i\epsilon_{ijk}\sigma_k)\phi \\ -\phi^\dagger(\sigma_i\sigma_j\delta_{ij} + i\epsilon_{ijk}\sigma_k)(\nabla_j\phi)\Big] \tag{5.21}$$

$$\mathrm{J}_{\mathrm{Levy-Leblond}_i} = \frac{i\hbar}{2m}\Big[(\nabla_i\phi^\dagger)\phi - \phi^\dagger(\nabla_i\phi) \\ -i\epsilon_{jki}\phi^\dagger(\nabla_j\sigma_k\phi) - i\epsilon_{jki}(\nabla_j\phi^\dagger)\sigma_k\phi\Big] \tag{5.22}$$

$$\mathrm{J}_{\mathrm{Levy-Leblond}_i} = \frac{i\hbar}{2m}\left[(\nabla_i\phi^\dagger)\phi - \phi^\dagger(\nabla_i\phi)\right] + \frac{\hbar}{2m}\epsilon_{jki}\nabla_j(\phi^\dagger\sigma_k\phi) \tag{5.23}$$

$$\mathrm{J}_{\mathrm{Levy-Leblond}_i} = \frac{i\hbar}{2m}\left[(\nabla_i\phi^\dagger)\phi\right] - \phi^\dagger(\nabla_i\phi) + \frac{\hbar}{2m}\left[\nabla \times \phi^\dagger\sigma\phi\right]_i. \tag{5.24}$$

Our final result then is

$$\boxed{\mathbf{J}_{\mathrm{Levy-Leblond}} = \frac{i\hbar}{2m}\left[(\nabla\phi^\dagger)\phi - \phi^\dagger(\nabla\phi)\right] + \frac{\hbar}{2m}\nabla \times (\phi^\dagger\sigma\phi).} \tag{5.25}$$

$$\mathbf{J}_{\mathrm{Levy-Leblond}} = \mathbf{J}_{\mathrm{Pauli}} + \mathbf{J}_{\mathrm{spin}} \tag{5.26}$$

In this final result we can plainly discern the extra spin current term. As it appears here, it would seem that this result accomplishes the first research objective of a nonrelativistic derivation of spin current. The spin current term we have derived has the same coefficient and form as the result obtained through the nonrelativistic limit of the Dirac current in Section **??**. However, the very careful reader may not be so easily convinced. When we examine the spin current term, we notice that it is in the form of a curl. This implies that the spin current term has no significant role in the continuity equation since in this equation the current appears under a divergence. Thus, since the divergence of a curl is zero, any contribution of the spin current in the continuity equation will automatically be zero. How then can it be possible to derive a unique coefficient for the spin current term if at one point in the derivation, namely in the continuity equation, the spin current evaluates to zero? If indeed the contribution of the spin current term is zero, surely we can with impunity place any coefficient before the term and still satisfy the continuity equation. How can we be sure the coefficient we derived in (**??**) above is correct and unique?

In short, the answer to this question is that the constraints that define the unique coefficient of the spin current are found within the actual process of calculating the total probability current. Although, the continuity equation will allow any coefficient, the process of constructing the current determines the coefficient uniquely.

In order to better understand how these constraints arise, we must examine exactly how the spin current term is generated in the calculation. We can do this by working backwards through the derivation and tracing the origin of the spin current term.

The separation of the spin current term from the regular Pauli current occurs after line (**??**) which reads

$$\mathrm{J}_{\mathrm{Levy-Leblond}_i} = -\frac{i\hbar}{2m}[\phi^\dagger \sigma_i \sigma_j (\nabla_j \phi) - (\nabla_j \phi^\dagger)\sigma_j \sigma_i \phi]. \tag{5.27}$$

The two terms in this expression each contain an even and odd part with respect to permutation of the σ spin operators. When we split both these terms into their even and odd parts we obtain four terms. The two even parts, when taken together, form the Pauli current. The two odd parts, when taken together, form the spin current. With this understanding we can now address the question of why the coefficient of the spin current cannot be arbitrary.

The key is that the odd terms which combine to form the spin current are tied to the even parts that form the Pauli current. If we increase the coefficient of the spin current, then we are constrained to increase the coefficient of the even part to which it is bound. Any change in the spin current coefficient would cause the Pauli current coefficient to be altered also. But, the correct coefficient of the Pauli current is a unique value determined by the constraints of normalization. Thus, since the Pauli current coefficient is fixed, the spin current coefficient must also be fixed at a unique value.

The manner in which the correct coefficient appears in this derivation is not unique to the Levy-Leblond equation. Indeed, the same arguments presented here must be made for the derivation of the Dirac current used by Nowakowski. The identity known as the Gordon decomposition [**?**] demonstrates that the Dirac current, $\mathbf{J}_{\mathrm{Dirac}} = c(\Psi^\dagger \alpha \Psi)$, can be separated into two terms describing the regular convection current and the spin current. Under the divergence of the continuity equation, this Dirac spin current term is effectively zero, however, must be retained with a unique coefficient in order to satisfy other constraints imposed elsewhere in the derivation.

We have taken the Levy-Leblond equation, a nonrelativistic expression,

Figure 5.1: The probability current for the nonrelativistic Levy-Leblond equation includes an extra spin current term. This result agrees with the nonrelativistic limit of the Dirac current. The spin current term is not uniquely defined when we calculate the probability current using the Pauli equation.

Figure 5.2: Spin current has the form of a curl.

and calculated its corresponding probability current. As illustrated in Figure **??**, the result includes the extra spin current term with the correct and unique coefficient. Thus, spin current can be derived without appealing to relativistic quantum mechanics. Whereas Nowakowski uses the Lorentz symmetry to introduce the needed constraints, Levy-Leblond is able to retrieve these constraints by applying Galilean symmetry only. We conclude that the spin current is a nonrelativistic effect that must be included in the expression for the nonrelativistic probability current of particles with spin. This is a significant result since standard textbook treatments of the probability current for the Pauli equation typically neglect the contribution of spin to the overall current [**?**]. The results from this calculation effectively resolve the first objective of this research.

5.2 Properties of Spin Current

The fact that spin contributes to the total probability current in a nonrelativistic regime merits a study of its properties and behavior. It will be interesting to determine whether this seldom-spoken-of spin current produces any experimentally measurable contributions to the probability current.

As has been previously noted, the spin current term,

$$\mathbf{J}_{\text{spin}} = \frac{\hbar}{2m} \nabla \times (\psi^\dagger \sigma \psi), \tag{5.28}$$

has the form of a curl. This implies that the spin current cannot diverge from an origin, but may only swirl about it as depicted in Figure **??**.

The curl property of spin current precludes it from contributing to the global flow of probability density from one location to another. Thus, spin current cannot contribute to the momentum of the particle. Nevertheless, we may not conclude from this observation that the spin current has no physical significance. The swirling spin current still provides a nontrivial measurable contribution, as any unlucky soul who has fallen into a whirlpool will attest!

Further properties of the spin current are more easily exposed by an examination of its behavior for a particular physical configuration. A calculation of the spin current for the specific case of an electron within a homogeneous magnetic field proves to be an informative exercise in this regard.

5.2.1 An electron in a homogeneous magnetic field

We shall be most interested in comparing the probability current of the Pauli equation, where the spin current term is absent, with the probability current of the Levy-Leblond equation which includes the extra spin current term. The comparison between the two results will illuminate the effects contributed by the spin current. If the effects of spin current are significant, an experiment can be devised to measure and verify them.

Obtaining the probability currents for an electron in a homogeneous magnetic field requires that we first derive solutions to the Pauli and Levy-Leblond wave equations for the corresponding potential. L. Landau published the first solution to this problem for the Schrödinger equation in 1930.[3] We can use his results to facilitate our solution of this configuration for the Pauli and Levy-Leblond equations. Once the correct wave functions have been obtained, we can proceed to calculate the exact probability currents using the expressions we have previously derived.

Solution to the Pauli equation

Our aim here is to solve the Pauli equation for an electron in a homogeneous magnetic field. We begin with the Pauli Equation,

$$\left[\frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 - \frac{e\hbar}{2m}\sigma.\mathbf{B}\right]\psi = E\psi. \tag{5.29}$$

In this expression, ψ represents a two-component wave function:

$$\psi \equiv \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}. \tag{5.30}$$

In order to ensure a homogeneous magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$, we choose the vector potential to be

[3] See Appendix **??** Solution to the Schrödinger equation for an electron in a homogeneous magnetic field. A solution to the relativistic problem was provided by I. Rabi in 1928 [**?**]

$$\mathbf{A} = B_0 x\hat{\mathbf{y}}. \tag{5.31}$$

Substituting for $\mathbf{B}$ and $\mathbf{p}$ in the Pauli equation yields

$$\left[\frac{1}{2m}(\frac{\hbar}{i}\nabla - e\mathbf{A})^2 - \frac{e\hbar}{2m}\begin{pmatrix} B_0 & 0 \\ 0 & -B_0 \end{pmatrix}\right]\begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = E\begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}, \tag{5.32}$$

or, by rearranging terms,

$$\left[\frac{1}{2m}(\frac{\hbar}{i}\nabla - e\mathbf{A})^2\right]\begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = \left[E + \frac{e\hbar}{2m}\begin{pmatrix} B_0 & 0 \\ 0 & -B_0 \end{pmatrix}\right]\begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}. \tag{5.33}$$

We can pick off the top and bottom equations to obtain two separate equations. The two equations are

$$\frac{1}{2m}(\frac{\hbar}{i}\nabla - e\mathbf{A})^2\psi_a = (E + \frac{e\hbar}{2m}B_0)\psi_a \tag{5.34}$$

$$\frac{1}{2m}(\frac{\hbar}{i}\nabla - e\mathbf{A})^2\psi_b = (E - \frac{e\hbar}{2m}B_0)\psi_b. \tag{5.35}$$

These two equations have exactly the same form as the Schrödinger equation for which we have already obtained solutions by Landau.[4] The Schrödinger equation is

$$\frac{1}{2m}(\frac{\hbar}{i}\nabla - e\mathbf{A})^2\psi = E\psi. \tag{5.36}$$

Thus, we can use the solutions already obtained if we let

$$E_\uparrow = E + \frac{e\hbar}{2m}B_0 \tag{5.37}$$

for the spin up case, and

$$E_\downarrow = E - \frac{e\hbar}{2m}B_0 \tag{5.38}$$

for the spin down case.

Our final energies for the Pauli equation are

[4]See Appendix **??** for Landau's solution of the Schrödinger equation for an electron in a homogeneous magnetic field.

$$
{}_{\uparrow}E_{n,k_z} = (n + \frac{1}{2})\hbar w_c + \frac{\hbar^2}{2m}k_z^2 + \frac{e\hbar}{2m}B_0 \tag{5.39}
$$

for spin up, and

$$
{}_{\downarrow}E_{n,k_z} = (n + \frac{1}{2})\hbar w_c + \frac{\hbar^2}{2m}k_z^2 - \frac{e\hbar}{2m}B_0 \tag{5.40}
$$

for spin down where n is the principal quantum number arising from the quantization of energy.

Evidently, the spatial part of the wave functions are identical to those which we have already found for the Schrödinger equation only that now there is a different energy dependent on the spin of the particle.

The wave function for spin up is

$$
{}_{\uparrow}\psi_n = \begin{pmatrix} \psi_n \\ 0 \end{pmatrix}, \tag{5.41}
$$

and for spin down

$$
{}_{\downarrow}\psi_n = \begin{pmatrix} 0 \\ \psi_n \end{pmatrix}, \tag{5.42}
$$

where

$$
\psi_n = (\frac{m\omega_c}{\pi\hbar})^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\xi^2/2+ik_y y+ik_z z} \tag{5.43}
$$

$$
\omega_c = \frac{eB_0}{m} \qquad \xi = \sqrt{\frac{m\omega_c}{\hbar}}(x - x_0) \qquad x_0 = \frac{\hbar k_y}{eB_0}, \tag{5.44}
$$

and $H_n(\xi)$ are the Hermite polynomials.

Having obtained the wave equations, we may now proceed to calculate the probability current for the Pauli equation. The expression for calculating the Pauli probability current in an electromagnetic field is given by[5]

$$
\mathbf{J}_{\text{Pauli}} = \frac{i\hbar}{2m}\Big[(\nabla\psi^\dagger)\psi - \psi^\dagger(\nabla\psi)\Big] - \frac{e}{m}(\psi^\dagger\mathbf{A}\psi). \tag{5.45}
$$

[5] See Appendix **??** Probability current for the Pauli equation in an electromagnetic field.

Evaluating this expression reveals that the currents for both spin up and spin down are identical:

$$
{}_{\uparrow\downarrow}\mathbf{J}_{\text{Pauli}} = \frac{1}{2^n n!\sqrt{\pi}} e^{-\xi^2} [H_n(\xi)]^2 \left[0, -\omega_c \xi, \sqrt{\frac{\hbar\omega_c}{m}} k_z\right]. \tag{5.46}
$$

Solution to the Levy-Leblond equation

As we shall see anon, our solution to the Pauli equation expedites our solution of the Levy-Leblond equation. Again, we choose the vector potential to be

$$
\mathbf{A} = B_0 x \hat{\mathbf{y}}. \tag{5.47}
$$

Next we write the Levy-Leblond Equation,

$$
\begin{aligned}
E\phi - c(\sigma.\mathbf{p})\chi &= 0 \\
-c(\sigma.\mathbf{p})\phi + 2mc^2\chi &= 0.
\end{aligned} \tag{5.48}
$$

We take this equation into the electromagnetic field using minimal coupling, where $\mathbf{p} \rightarrow \mathbf{p} - e\mathbf{A}$, and $E \rightarrow E + e\Phi$. With these substitutions, the wave equation becomes

$$
\begin{aligned}
(E + e\Phi)\phi - c\left[\sigma.(\mathbf{p} - e\mathbf{A})\right]\chi &= 0 \\
-c\left[\sigma.(\mathbf{p} - e\mathbf{A})\right]\phi + 2mc^2\chi &= 0.
\end{aligned} \tag{5.49}
$$

Solving for χ in the second line,

$$
\chi = \frac{1}{2mc}\Big[\sigma.(\mathbf{p} - e\mathbf{A})\Big]\phi, \tag{5.50}
$$

and substituting in the first yields

$$
(E + e\Phi)\phi - c\sigma.(\mathbf{p} - e\mathbf{A})\left[\frac{1}{2mc}\sigma.(\mathbf{p} - e\mathbf{A})\right] = 0 \tag{5.51}
$$

$$
E\phi = \frac{1}{2m}\Big[\sigma.(\mathbf{p} - e\mathbf{A})\Big]^2\phi - e\Phi\phi. \tag{5.52}
$$

After expanding out the terms, it is a straight forward exercise to demonstrate that this expression simplifies to the Pauli equation.[6] We have effectively uncoupled the two first-order equations of the Levy-Leblond equation to produce a single second-order equation. Having already solved the Pauli equation, we can immediately write the solutions to the Levy-Leblond equation.

Our final energies are

$$E_{\uparrow,n,k_z} = (n + \frac{1}{2})\hbar w_c + \frac{\hbar^2}{2m}k_z^2 - \frac{e\hbar}{2m}B_0 \tag{5.53}$$

for spin up, and

$$E_{\downarrow,n,k_z} = (n + \frac{1}{2})\hbar w_c + \frac{\hbar^2}{2m}k_z^2 + \frac{e\hbar}{2m}B_0 \tag{5.54}$$

for spin down.

The wave function for spin up is

$$\phi_{\uparrow n} = \begin{pmatrix} \phi_n \\ 0 \end{pmatrix}, \tag{5.55}$$

and for spin down

$$\phi_{\downarrow n} = \begin{pmatrix} 0 \\ \phi_n \end{pmatrix} \tag{5.56}$$

where

$$\phi_n(x, y, z) = (\frac{m\omega_c}{\pi\hbar})^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\xi^2/2+ik_y y+ik_z z} \tag{5.57}$$

$$\omega_c = \frac{eB_0}{m} \qquad \xi = \sqrt{\frac{m\omega_c}{\hbar}}(x - x_0) \qquad x_0 = \frac{\hbar k_y}{eB_0}. \tag{5.58}$$

With the exact solutions for ϕ in hand, it will be a straightforward exercise to find χ using the equation in (**??**). However, our expression for the Levy-Leblond probability current in an electromagnetic field,[7]

[6] See Appendix **??** Derivation of the Pauli equation beginning at line (**??**).

[7] See Appendix **??** Probability current for the Levy-Leblond equation for an electron in an electromagnetic field.

$$\mathbf{J}_{\text{Levy-LeblondEM}} = \frac{i\hbar}{2m}\left[(\nabla\phi^\dagger)\phi - \phi^\dagger(\nabla\phi)\right] - \frac{e}{m}(\phi^\dagger\mathbf{A}\phi) + \frac{\hbar}{2m}\nabla\times(\phi^\dagger\sigma\phi), \tag{5.59}$$

is in terms of ϕ only. Since our main purpose is to analyze the probability current, we shall proceed to the probability current calculation without delay. As we do so, it will be interesting to observe closely the contribution of the spin current term $\frac{\hbar}{2m}\nabla\times(\phi^\dagger\sigma\phi)$.

We shall evaluate each component of the current vector in turn, noting that $A_x = 0$, $A_y = B_0 x$, and $A_z = 0$. We begin with J_x for the spin up case. We shall omit the subscript n, since these results are valid for all n.

$$_\uparrow\mathrm{J}_x = \frac{i\hbar}{2m}\left[(\partial_x\phi_\uparrow^\dagger)\phi_\uparrow - \phi_\uparrow^\dagger(\partial_x\phi_\uparrow)\right] + \frac{\hbar}{2m}\left[\partial_y(\phi_\uparrow^\dagger\sigma_z\phi_\uparrow) - \partial_z(\phi_\uparrow^\dagger\sigma_y\phi_\uparrow)\right] \tag{5.60}$$

$$\begin{aligned} _\uparrow\mathrm{J}_x &= \frac{i\hbar}{2m}\left[(\partial_x\phi^*)\phi - \phi^*(\partial_x\phi)\right] \qquad (5.61)\\ &+\frac{\hbar}{2m}\left[\partial_y\left[\begin{pmatrix}\phi^* & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\phi\\ 0\end{pmatrix}\right] - \partial_z\left[\begin{pmatrix}\phi^* & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\phi\\ 0\end{pmatrix}\right]\right] \end{aligned}$$

Performing the matrix multiplication reduces this to

$$_\uparrow\mathrm{J}_x = \underbrace{\frac{i\hbar}{2m}\left[(\partial_x\phi^*)\phi - \phi^*(\partial_x\phi)\right]}_{\text{Pauli current}} + \underbrace{\frac{\hbar}{2m}\left[\partial_y(\phi^*\phi)\right]}_{\text{spin current}}. \tag{5.62}$$

Both the Pauli current term and the spin current term in this equation are identically zero, leaving us with the trivial answer

$$_\uparrow\mathrm{J}_x = 0. \tag{5.63}$$

Now to find $_\uparrow\mathrm{J}_y$.

$$_\uparrow\mathrm{J}_y = \frac{i\hbar}{2m}\left[(\partial_y\phi_\uparrow^\dagger)\phi_\uparrow - \phi_\uparrow^\dagger(\partial_y\phi_\uparrow)\right] - \frac{eB_0}{m}x(\phi_\uparrow^\dagger\phi_\uparrow) + \frac{\hbar}{2m}\left[\partial_z(\phi_\uparrow^\dagger\sigma_x\phi_\uparrow) - \partial_x(\phi_\uparrow^\dagger\sigma_z\phi_\uparrow)\right] \tag{5.64}$$

$$
\begin{aligned}
{}_{\uparrow}\mathrm{J}_y \;&=\; \frac{i\hbar}{2m}\left[(\partial_y\phi^*)\phi - \phi^*(\partial_y\phi)\right] - \frac{eB_0}{m}x(\phi^*\phi) \qquad (5.65)\\
&+\frac{\hbar}{2m}\left[\partial_z\left[\begin{pmatrix}\phi^* & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\phi\\ 0\end{pmatrix}\right] - \partial_x\left[\begin{pmatrix}\phi^* & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\phi\\ 0\end{pmatrix}\right]\right]
\end{aligned}
$$

Performing the matrix multiplication reduces this to

$$
{}_{\uparrow}\mathrm{J}_y = \underbrace{\frac{i\hbar}{2m}\left[(\partial_y\phi^*)\phi - \phi^*(\partial_y\phi)\right] - \frac{eB_0}{m}x(\phi^*\phi)}_{\text{Pauli current}} - \underbrace{\frac{\hbar}{2m}\left[\partial_x(\phi^*\phi)\right]}_{\text{spin current}}. \qquad (5.66)
$$

Notice that the contribution from the spin current term in this equation is non-zero. After evaluating the derivatives, we can pull out a common factor of $\frac{e^{-\xi^2}[H_n(\xi)]^2}{2^n n!\sqrt{\pi}}$ from each term:

$$
\begin{aligned}
{}_{\uparrow}\mathrm{J}_y \;&=\; \frac{e^{-\xi^2}[H_n(\xi)]^2}{2^n n!\sqrt{\pi}}\Bigg[\overbrace{\sqrt{\frac{\hbar\omega_c}{m}}k_y - \sqrt{\frac{m\omega_c}{\hbar}}\left(\frac{eB_0}{m}\right)x}^{\text{Pauli current}} \qquad (5.67)\\
&\underbrace{-\sqrt{\frac{\omega_c}{m\hbar}}\left(2n\sqrt{m\omega_c\hbar}\frac{H_{n-1}(\xi)}{H_n(\xi)} - m\omega_c x + m\omega_c x_0\right)}_{\text{spin current}}\Bigg].
\end{aligned}
$$

Now we must remember the relations

$$
\omega_c = \frac{eB_0}{m} \qquad \xi = \sqrt{\frac{m\omega_c}{\hbar}}(x - x_0) \qquad x_0 = \frac{\hbar k_y}{eB_0}. \qquad (5.68)
$$

Substituting for k_y in the first term, $\left(\frac{eB_0}{m}\right)$ in the second, and expanding out the third yields

$$
\begin{aligned}
{}_{\uparrow}\mathrm{J}_y \;&=\; \frac{e^{-\xi^2}[H_n(\xi)]^2}{2^n n!\sqrt{\pi}}\Bigg[\sqrt{\frac{\omega_c}{m\hbar}}eB_0x_0 - \sqrt{\frac{m\omega_c}{\hbar}}\omega_c x \qquad (5.69)\\
&-2n\omega_c\frac{H_{n-1}(\xi)}{H_n(\xi)} + \sqrt{\frac{m\omega_c}{\hbar}}\omega_c x - \sqrt{\frac{m\omega_c}{\hbar}}\omega_c x_0)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
{}_{\uparrow}\mathrm{J}_y &= \frac{e^{-\xi^2}[H_n(\xi)]^2}{2^n n!\sqrt{\pi}}\left[\sqrt{\frac{m\omega_c}{\hbar}}\left(\frac{eB_0}{m}\right)x_0 - \sqrt{\frac{m\omega_c}{\hbar}}\omega_c x\right. \\
&\qquad \left. -2n\omega_c\frac{H_{n-1}(\xi)}{H_n(\xi)} + \sqrt{\frac{m\omega_c}{\hbar}}\omega_c x - \sqrt{\frac{m\omega_c}{\hbar}}\omega_c x_0)\right]
\end{aligned} \tag{5.70}
$$

$$
\begin{aligned}
{}_{\uparrow}\mathrm{J}_y &= \frac{e^{-\xi^2}[H_n(\xi)]^2}{2^n n!\sqrt{\pi}}\left[\overbrace{-\sqrt{\frac{m\omega_c}{\hbar}}\omega_c(x-x_0)}^{\text{Pauli current}}\right. \\
&\qquad \left. -\underbrace{2n\omega_c\frac{H_{n-1}(\xi)}{H_n(\xi)} + \sqrt{\frac{m\omega_c}{\hbar}}\omega_c(x-x_0)}_{\text{spin current}}\right].
\end{aligned} \tag{5.71}
$$

At this point we pause to observe an interesting interaction between the Pauli current and the spin current. The spin current contributes a term that exactly cancels the Pauli current. All that survives is a single remaining term from the spin current.

$$
{}_{\uparrow}\mathrm{J}_y = \frac{e^{-\xi^2}[H_n(\xi)]^2}{2^n n!\sqrt{\pi}}\left[-2n\omega_c\frac{H_{n-1}(\xi)}{H_n(\xi)}\right] \tag{5.72}
$$

Finally then,

$$
{}_{\uparrow}\mathrm{J}_y = -\frac{1}{2^n n!\sqrt{\pi}}e^{-\xi^2}[H_n(\xi)]^2\left[2n\omega_c\frac{H_{n-1}(\xi)}{H_n(\xi)}\right]. \tag{5.73}
$$

Lastly, we must find ${}_{\uparrow}\mathrm{J}_z$.

$$
{}_{\uparrow}\mathrm{J}_z = \underbrace{\frac{i\hbar}{2m}\left[(\partial_z\phi^\dagger)\phi - \phi^\dagger(\partial_z\phi)\right]}_{\text{Pauli current}} + \underbrace{\frac{\hbar}{2m}\left[\partial_x(\phi^\dagger\sigma_y\phi) - \partial_y(\phi^\dagger\sigma_x\phi)\right]}_{\text{spin current}} \tag{5.74}
$$

$$
\begin{aligned}
{}_{\uparrow}\mathrm{J}_z &= \frac{i\hbar}{2m}\left[(\partial_z\phi^*)\phi - \phi^*(\partial_z\phi)\right] \\
&\quad + \frac{\hbar}{2m}\left[\partial_x\left[\begin{pmatrix}\phi^* & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\phi\\ 0\end{pmatrix}\right] - \partial_y\left[\begin{pmatrix}\phi^* & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\phi\\ 0\end{pmatrix}\right]\right]
\end{aligned} \tag{5.75}
$$

Performing the matrix multiplication reveals that the entire spin current term is zero, leaving us with the same result as the Pauli current,

$$_{\uparrow}\mathrm{J}_z = \frac{i\hbar}{2m}\left[(\partial_z\phi^*)\phi - \phi^*(\partial_z\phi)\right]. \tag{5.76}$$

This evaluates to

$$_{\uparrow}\mathrm{J}_z = \frac{1}{2^n n!\sqrt{\pi}} e^{-\xi^2}[H_n(\xi)]^2\sqrt{\frac{\hbar\omega_c}{m}}k_z \tag{5.77}$$

corresponding to the unconfined motion of the electron in the $\hat{\mathbf{z}}$ direction.

Our final result for the spin up current vector is

$$_{\uparrow}\mathbf{J}_{\mathrm{Levy-Leblond}} = \frac{1}{2^n n!\sqrt{\pi}} e^{-\xi^2}[H_n(\xi)]^2\left[0\ ,\ -2n\omega_c\frac{H_{n-1}(\xi)}{H_n(\xi)}\ ,\ \sqrt{\frac{\hbar\omega_c}{m}}k_z\right]. \tag{5.78}$$

Now we must evaluate the Levy-Leblond probability current for the spin down case where

$$\phi_{\downarrow} = \begin{pmatrix} 0 \\ \phi \end{pmatrix}. \tag{5.79}$$

Again we use the probability current equation,

$$\mathbf{J}_{\mathrm{Levy-LeblondEM}} = \frac{i\hbar}{2m}\left[(\nabla\phi^\dagger)\phi - \phi^\dagger(\nabla\phi)\right] - \frac{e}{m}(\phi^\dagger\mathbf{A}\phi) + \frac{\hbar}{2m}\nabla\times(\phi^\dagger\sigma\phi), \tag{5.80}$$

Begin by looking at the x-component,

$$_{\downarrow}\mathrm{J}_x = \frac{i\hbar}{2m}\left[(\partial_x\phi^\dagger\downarrow)\phi\downarrow - \phi^\dagger\downarrow(\partial_x\phi\downarrow)\right] + \frac{\hbar}{2m}\left[\partial_y(\phi^\dagger\downarrow\sigma_z\phi\downarrow) - \partial_z(\phi^\dagger\downarrow\sigma_y\phi\downarrow)\right] \tag{5.81}$$

$$
\begin{aligned}
{}_{\downarrow}\mathrm{J}_x &= \frac{i\hbar}{2m}\left[(\partial_x\phi^*)\phi - \phi^*(\partial_x\phi)\right] \\
&+ \frac{\hbar}{2m}\left[\partial_y\left[\begin{pmatrix} 0 & \phi^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ \phi \end{pmatrix}\right] - \partial_z\left[\begin{pmatrix} 0 & \phi^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \phi \end{pmatrix}\right]\right].
\end{aligned} \tag{5.82}
$$

Performing the matrix multiplication reduces this to

$$
{}_{\downarrow}\mathrm{J}_x = \underbrace{\frac{i\hbar}{2m}\left[(\partial_x\phi^*)\phi - \phi^*(\partial_x\phi)\right]}_{\text{Pauli current}} - \underbrace{\frac{\hbar}{2m}\left[\partial_y(\phi^*\phi)\right]}_{\text{spin current}}. \tag{5.83}
$$

Again, just as in the spin up case, both terms here are identically zero, so

$$
{}_{\downarrow}\mathrm{J}_x = 0. \tag{5.84}
$$

Now to find ${}_{\downarrow}\mathrm{J}_y$.

$$
{}_{\downarrow}\mathrm{J}_y = \frac{i\hbar}{2m}\left[(\partial_y\phi^\dagger)\phi - \phi^\dagger(\partial_y\phi)\right] - \frac{eB_0}{m}x(\phi^\dagger\phi) + \frac{\hbar}{2m}\left[\partial_z(\phi^\dagger\sigma_x\phi) - \partial_x(\phi^\dagger\sigma_z\phi)\right] \tag{5.85}
$$

$$
\begin{aligned}
{}_{\downarrow}\mathrm{J}_y &= \frac{i\hbar}{2m}\left[(\partial_y\phi^*)\phi - \phi^*(\partial_y\phi)\right] - \frac{eB_0}{m}x(\phi^*\phi) \\
&+ \frac{\hbar}{2m}\left[\partial_z\left[\begin{pmatrix} 0 & \phi^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \phi \end{pmatrix}\right] - \partial_x\left[\begin{pmatrix} 0 & \phi^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ \phi \end{pmatrix}\right]\right]
\end{aligned} \tag{5.86}
$$

Performing the matrix multiplication reduces this to

$$
{}_{\downarrow}\mathrm{J}_y = \underbrace{\frac{i\hbar}{2m}\left[(\partial_y\phi^*)\phi - \phi^*(\partial_y\phi)\right] - \frac{eB_0}{m}x(\phi^*\phi)}_{\text{Pauli current}} + \underbrace{\frac{\hbar}{2m}\left[\partial_x(\phi^*\phi)\right]}_{\text{spin current}}. \tag{5.87}
$$

Notice again that the contribution from the spin current term is non-zero. In each term we can pull out a common factor of $\frac{e^{-\xi^2}[H_n(\xi)]^2}{2^n n!\sqrt{\pi}}$ to find

$$
{}_{\downarrow}\mathrm{J}_y = \frac{e^{-\xi^2}[H_n(\xi)]^2}{2^n n!\sqrt{\pi}}\Bigg[\overbrace{\sqrt{\frac{\hbar\omega_c}{m}}k_y - \sqrt{\frac{m\omega_c}{\hbar}}\left(\frac{eB_0}{m}\right)x}^{\text{Pauli current}} \tag{5.88}
$$

$$
\underbrace{+\sqrt{\frac{\omega_c}{m\hbar}}\left(2n\sqrt{m\omega_c\hbar}\frac{H_{n-1}(\xi)}{H_n(\xi)} - m\omega_c x + m\omega_c x_0)\right)}_{\text{spin current}}\Bigg].
$$

Now we use the relations

$$
\omega_c = \frac{eB_0}{m} \qquad \xi = \sqrt{\frac{m\omega_c}{\hbar}}(x - x_0) \qquad x_0 = \frac{\hbar k_y}{eB_0}. \tag{5.89}
$$

Substituting for k_y in the first term, $\left(\frac{eB_0}{m}\right)$ in the second, and expanding out the third gives

$$
{}_{\downarrow}\mathrm{J}_y = \frac{e^{-\xi^2}[H_n(\xi)]^2}{2^n n!\sqrt{\pi}}\Bigg[\sqrt{\frac{\omega_c}{m\hbar}}eB_0x_0 - \sqrt{\frac{m\omega_c}{\hbar}}\omega_c x \tag{5.90}
$$

$$
+2n\omega_c\frac{2nH_{n-1}(\xi)}{H_n(\xi)} - \sqrt{\frac{m\omega_c}{\hbar}}\omega_c x + \sqrt{\frac{m\omega_c}{\hbar}}\omega_c x_0)\Bigg]
$$

$$
{}_{\downarrow}\mathrm{J}_y = \frac{e^{-\xi^2}[H_n(\xi)]^2}{2^n n!\sqrt{\pi}}\Bigg[\sqrt{\frac{m\omega_c}{\hbar}}\left(\frac{eB_0}{m}\right)x_0 - \sqrt{\frac{m\omega_c}{\hbar}}\omega_c x \tag{5.91}
$$

$$
+2n\omega_c\frac{H_{n-1}(\xi)}{H_n(\xi)} - \sqrt{\frac{m\omega_c}{\hbar}}\omega_c x + \sqrt{\frac{m\omega_c}{\hbar}}x_0)\Bigg]
$$

$$
{}_{\downarrow}\mathrm{J}_y = \frac{e^{-\xi^2}[H_n(\xi)]^2}{2^n n!\sqrt{\pi}}\Bigg[\overbrace{-\sqrt{\frac{m\omega_c}{\hbar}}\omega_c(x - x_0)}^{\text{Pauli current}} \tag{5.92}
$$

$$
+\underbrace{2n\omega_c\frac{H_{n-1}(\xi)}{H_n(\xi)} - \sqrt{\frac{m\omega_c}{\hbar}}\omega_c(x - x_0)}_{\text{spin current}}\Bigg].
$$

At this point in the calculation it is informative to examine again the behaviour of the spin current term. The spin current produces a term which exactly duplicates the Pauli current term. An additional term from the spin current also survives.

After gathering terms, we have

$$_{\downarrow}\mathrm{J}_y = \frac{1}{2^n n!\sqrt{\pi}} e^{-\xi^2} [H_n(\xi)]^2 \left[-2\sqrt{\frac{m\omega_c}{\hbar}} \omega_c (x - x_0) + 2n\omega_c \frac{H_{n-1}(\xi)}{H_n(\xi)} \right] \quad (5.93)$$

$$_{\downarrow}\mathrm{J}_y = \frac{1}{2^n n!\sqrt{\pi}} e^{-\xi^2} [H_n(\xi)]^2 \left[2n\omega_c \frac{nH_{n-1}(\xi)}{H_n(\xi)} - 2\xi \right]. \quad (5.94)$$

Lastly, the z-component:

$$_{\downarrow}\mathrm{J}_z = \frac{i\hbar}{2m} \left[(\partial_z \phi^\dagger)\phi - \phi^\dagger(\partial_z \phi) \right] + \frac{\hbar}{2m} \left[\partial_x(\phi^\dagger \sigma_y \phi) - \partial_y(\phi^\dagger \sigma_x \phi) \right] \quad (5.95)$$

$$\begin{aligned} _{\downarrow}\mathrm{J}_z &= \frac{i\hbar}{2m} \left[(\partial_z \phi^*)\phi - \phi^*(\partial_z \phi) \right] \quad (5.96) \\ &+ \frac{\hbar}{2m} \left[\partial_x \left[\begin{pmatrix} 0 & \phi^* \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \phi \end{pmatrix} \right] - \partial_y \left[\begin{pmatrix} 0 & \phi^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \phi \end{pmatrix} \right] \right]. \end{aligned}$$

Performing the matrix multiplication reveals that the second term is zero. This result signifies that spin current makes no contribution to the z-component of the current. We are left with

$$_{\downarrow}\mathrm{J}_z = \frac{i\hbar}{2m} \left[(\partial_z \phi^*)\phi - \phi^*(\partial_z \phi) \right] \quad (5.97)$$

which evaluates to

$$_{\downarrow}\mathrm{J}_z = \frac{1}{2^n n!\sqrt{\pi}} e^{-\xi^2} [H_n(\xi)]^2 \sqrt{\frac{\hbar\omega_c}{m}} k_z. \quad (5.98)$$

Our final result for the spin down case current vector is

$$
{}_{\downarrow}\mathbf{J}_{\text{LevyLeblond}} = \frac{1}{2^n n!\sqrt{\pi}} e^{-\xi^2}[H_n(\xi)]^2 \left[0\ ,\ 2n\omega_c \frac{H_{n-1}(\xi)}{H_n(\xi)} - 2\xi\ ,\ \sqrt{\frac{\hbar\omega_c}{m}} k_z \right]. \tag{5.99}
$$

Remember that the current for the spin up case was

$$
{}_{\uparrow}\mathbf{J}_{\text{LevyLeblond}} = \frac{1}{2^n n!\sqrt{\pi}} e^{-\xi^2}[H_n(\xi)]^2 \left[0\ ,\ -2n\omega_c \frac{H_{n-1}(\xi)}{H_n(\xi)}\ ,\ \sqrt{\frac{\hbar\omega_c}{m}} k_z \right]. \tag{5.100}
$$

We will compare these with the current for the Pauli equation,

$$
{}_{\uparrow\downarrow}\mathbf{J}_{\text{Pauli}} = \frac{1}{2^n n!\sqrt{\pi}} e^{-\xi^2}[H_n(\xi)]^2 [0, -\omega_c\xi, \sqrt{\frac{\hbar\omega_c}{m}} k_z], \tag{5.101}
$$

(the same for spin up and spin down) in order to analyze the contribution of the spin current.

5.2.2 Analysis of spin current effects

We have observed during the preceding calculations that the spin current interacts with the Pauli current in an interesting manner. Only in the y-component of the probability current does spin current generate a nontrivial contribution. Accordingly, we shall focus our attention on this component of the probability current. For each of the wave equations they are

$$
\mathrm{J}_{\text{Pauli}_y} = -\frac{1}{2^n n!\sqrt{\pi}} e^{-\xi^2}[H_n(\xi)]^2 \sqrt{\frac{m\omega_c}{\hbar}} \omega_c (x - x_0), \tag{5.102}
$$

$$
{}_{\uparrow}\mathrm{J}_{\text{Levy-Leblond}_y} = -\frac{1}{2^n n!\sqrt{\pi}} e^{-\xi^2}[H_n(\xi)]^2 \left[2n\omega_c \frac{H_{n-1}(\xi)}{H_n(\xi)} \right], \tag{5.103}
$$

and

$$ {}_{\downarrow}\mathrm{J}_{\mathrm{Levy-Leblond}_y} = -\frac{1}{2^n n!\sqrt{\pi}}[H_n(\xi)]^2 e^{-\xi^2}\left[2\sqrt{\frac{m\omega_c}{\hbar}}(x-x_0) + 2n\omega_c\frac{H_{n-1}(\xi)}{H_n(\xi)}\right]. \tag{5.104} $$

The y-components of each probability current are summarized in Table **??**. Large and unwieldy factors have been renamed as uppercase letters A, B, and C:

$$ \mathrm{A} = -\frac{1}{2^n n!\sqrt{\pi}} e^{-\xi^2}[H_n(\xi)]^2\sqrt{\frac{m\omega_c}{\hbar}}\omega_c x, \tag{5.105} $$

$$ \mathrm{B} = \frac{1}{2^n n!\sqrt{\pi}} e^{-\xi^2}[H_n(\xi)]^2\sqrt{\frac{m\omega_c}{\hbar}}\omega_c x_0, \tag{5.106} $$

$$ \mathrm{C} = \frac{1}{2^n n!\sqrt{\pi}} e^{-\xi^2}[H_n(\xi)]^2\left[2n\omega_c\frac{H_{n-1}(\xi)}{H_n(\xi)}\right]. \tag{5.107} $$

This substitution allows for a clearer analysis of the inter-relationships between the various terms that contribute to the total current. The table illustrates how the Pauli and spin currents combine together to produce the total probability current.

The information presented in Table **??** proves most insightful for establishing the properties of spin current. First notice that, whereas the Pauli current is indifferent towards the up or down orientation of the spin, the spin current of the Levy-Leblond equation differentiates between the two spin states. As we may have intuitively conjectured at the outset, the contribution of spin current for spin up is exactly equal but opposite to the contribution for spin down. The Pauli current does not contain sufficient information about spin to discriminate between the two spin states. We notice that the total probability current for the Pauli equation is an exact average of the total probability currents for the spin up and spin down cases of the Levy Leblond equation. It appears that, rather than provide specific information on each spin state, the Pauli current prefers to give a single generalized statement.

A further intriguing observation is that the contribution from the spin current is not entirely arbitrary and disassociated from the Pauli current. Rather, it is connected to the Pauli current in such a way that it contains

Table 5.2: y-components of the probability currents for an electron in a homogeneous magnetic field.

Figure 5.3: Line graph of the y-component of the probability current for the Pauli equation for both spin up and spin down electrons.

Figure 5.4: Vector plot of the probability current for the Pauli equation for both spin up and spin down electrons.

similar components (A's and B's) that either combine with, or cancel corresponding terms in the Pauli current. For the case of spin up, terms in the expression for the spin current negate the Pauli current, while for spin down these spin current terms exactly duplicate the Pauli current.

Plots of the y-components of the Pauli and Levy-Leblond currents are shown graphically in Figures **??** - **??**. The line graphs provide an actual value for the magnitude of the y-component of the current for a given x, while the vector plots give a more physically intuitive feel for the behavior of the currents. The line graphs illustrate pictorially how a superposition or average of the Levy-Leblond currents for spin up and spin down results in the Pauli current.

Perhaps the most dramatic effect presented by the graphs is the physical reality of the spin current. The vector plots reveal that the spin current contribution provides an actual swirl as originally predicted. Since the electron carries a charge, this swirl due to the spin current would produce a flow of electrical current measurable by an electrical current meter. Presumably, this swirl of charge would create an effective current loop that would act as a magnetic dipole. A dipole would in turn experience a torque in the presence of the external magnetic field. A study of interaction effects due to the spin current will provide an interesting direction for further experimental research.

Figure 5.5: Line graph of the y-component of the probability current for the Levy-Leblond equation for spin up electrons.

Figure 5.6: Vector plot of the probability current for the Levy-Leblond equation for spin up electrons.

Figure 5.7: Line graph of the y-component of the probability current for the Levy-Leblond equation for spin down electrons.

Figure 5.8: Vector plot of the probability current for the Levy-Leblond equation for spin down electrons.

Chapter 6

Conclusion

The probability current for nonrelativistic particles with spin is seldom treated in accepted pedagogical formats of introductory quantum mechanics texts. The procedure for calculating probability current is most often presented with regard to the Schrödinger equation, an equation that describes non-relativistic particles without spin. When this standard procedure is applied to the Pauli equation, it yields an ambiguous outcome since the resultant probability current is not uniquely determined.

We have demonstrated that the unique expression for the probability current of nonrelativistic particles with spin must include an extra spin current term of the form $\frac{\hbar}{2m}\nabla \times (\psi^\dagger \sigma \psi)$. The inclusion of this extra term in the expression for the probability current indicates that the spin of the particle influences the flow of probability density.

Spin current is a nonrelativistic effect, and can be derived without appealing to relativity theory. In particular, we have derived the unique form of the spin current term using Levy-Leblond's nonrelativistic wave equation.

An analysis of the spin current for an electron in a homogeneous magnetic field reveals that the spin property, although it cannot contribute to the momentum of the particle, produces interesting interaction effects with the Pauli current. The swirl of the spin current introduces a nontrivial contribution that is potentially measurable by experiment. Specifically, the spin current contribution is dependent on the spin state of the electron.

Appendix A

Conventions and Formalism

A.1 Constants

e	The magnitude of the charge on an electron. Note that $e = \lvert e \rvert$.
$\hbar$	Planck's constant, h, divided by 2π.
c	The speed of light.

A.2 Explicit representations of α, β, γ, and σ matrices

The α and β matrices:

$$\alpha_k = \begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix} \qquad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{A.1}$$

The γ matrices:

$$\gamma_0 \equiv \beta \qquad \gamma_k \equiv \beta\alpha_k \tag{A.2}$$

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \tag{A.3}$$

The σ matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.4}$$

which obey the relations

$$\begin{aligned} \sigma_i \sigma_k &= \delta_{ik} + i\epsilon_{ikl}\sigma_l \\ \sigma_k \sigma_i &= \delta_{ki} - i\epsilon_{ikl}\sigma_l. \end{aligned} \tag{A.5}$$

Important Note: The matrices presented in this appendix are referred to in the body of the paper using their corresponding Greek letter α, β, γ, or σ. Whenever a Greek letter appears with an index, for example σ_y, we are referring only to a single matrix, which in this case is

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{A.6}$$

However, if we use a Greek letter without any index, for example α, we are referring to the multi-component object that contains all three matrices, α_1, α_2, and α_3.

Therefore,

$$\alpha.\mathbf{p} = \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_3. \tag{A.7}$$

A.3 Symbolic notation

A.3.1 Complex conjugation

When ψ is used to designate a scalar function (as in the Schrödinger equation), we represent the complex conjugate of ψ as ψ^*.

The Pauli equation involves two-component wave functions which we may express as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{A.8}$$

where ψ_1 and ψ_2 are scalar components of ψ. The complex conjugate of ψ in this instance is designated with a † symbol,

$$\psi^\dagger = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}^\dagger = \begin{pmatrix} \psi_1^* & \psi_2^* \end{pmatrix}. \tag{A.9}$$

This notation is also used with the multi-component wave functions of the Levy-Leblond and Dirac wave equations.

Throughout this paper we take the hermitian conjugate with respect to spin space only, and not with respect to coordinate space. For example,

$$(\nabla\psi)^\dagger = (\nabla\psi^\dagger). \tag{A.10}$$

A.3.2 Commutators

For any two arbitrary objects $\mathbf{A}$ and $\mathbf{B}$, we define the following two commutators:

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA} \tag{A.11}$$

$$[\mathbf{A}, \mathbf{B}]_+ = \mathbf{AB} + \mathbf{BA}. \tag{A.12}$$

Appendix B

Derivation of Wave Equations

B.1 Derivation of the Pauli equation in an electromagnetic field

We begin with the Schrödinger Equation:

$$\frac{\mathbf{p}^2}{2m} + V = E\psi. \tag{B.1}$$

Now with minimal coupling for a free particle in an electromagnetic field, and using σ to introduce spin, we let

$$\mathbf{p} \rightarrow (\mathbf{p} - e\mathbf{A}).\sigma, \qquad \text{and} \qquad E \rightarrow i\hbar\partial_t + e\Phi. \tag{B.2}$$

This yields

$$\frac{1}{2m}[(\mathbf{p} - e\mathbf{A}).\sigma]^2\psi = i\hbar\partial_t\psi + e\Phi. \tag{B.3}$$

We will concentrate for the moment on expanding the operator $[(\mathbf{p} - e\mathbf{A}).\sigma]^2$:

$$\begin{aligned}
[(\mathbf{p} - e\mathbf{A}).\sigma]^2 &= [(\mathbf{p} - e\mathbf{A}).\sigma][(\mathbf{p} - e\mathbf{A}).\sigma] && \text{(B.4)}\\
&= p_ip_i + e^2A_iA_i - e\sigma_i\sigma_jp_i(A_j) && \text{(B.5)}\\
&\quad -e\sigma_i\sigma_jA_jp_i - eA_ip_j\sigma_i\sigma_j &&\\
&= p^2 + e^2A^2 - e\Big[p_i(A_j)\sigma_i\sigma_j && \text{(B.6)}\\
&\quad +A_jp_i\sigma_i\sigma_j + A_ip_j\sigma_i\sigma_j\Big]. &&
\end{aligned}$$

Now use the fact that

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \tag{B.7}$$

to get

$$\begin{aligned}
[(\mathbf{p} - e\mathbf{A}).\sigma]^2 &= p^2 + e^2A^2 - e\Big[p_i(A_i) + 2A_ip_i \qquad \text{(B.8)}\\
&\quad + ip_iA_j\epsilon_{ijk}\sigma_k + iA_jp_i\epsilon_{ijk}\sigma_k + iA_ip_j\epsilon_{ijk}\sigma_k\Big]\\
&= p^2 + e^2A^2 - e\Big[p_i(A_i) + 2A_ip_i \qquad \text{(B.9)}\\
&\quad + ip_iA_j\epsilon_{ijk}\sigma_k - iA_ip_j\epsilon_{ijk}\sigma_k + iA_ip_j\epsilon_{ijk}\sigma_k\Big]\\
&= p^2 + e^2A^2 - e\left[\mathbf{p}.(\mathbf{A}) + 2\mathbf{A}.\mathbf{p} + ip_iA_j\epsilon_{ijk}\sigma_k\right] \qquad \text{(B.10)}\\
&= p^2 + e^2A^2 - e\mathbf{p}.\mathbf{A} - e\mathbf{A}.\mathbf{p} - ie\epsilon_{ijk}p_iA_j\sigma_k \qquad \text{(B.11)}\\
&= (\mathbf{p} - e\mathbf{A})^2 - ie\frac{\hbar}{i}(\nabla \times \mathbf{A})_k\sigma_k \qquad \text{(B.12)}\\
&= (\mathbf{p} - e\mathbf{A})^2 - e\hbar\mathbf{B}.\sigma \qquad \text{(B.13)}\\
&= (\mathbf{p} - e\mathbf{A})^2 - e\hbar\sigma.\mathbf{B}. \qquad \text{(B.14)}
\end{aligned}$$

Now we can substitute this back into (**??**) to get:

$$\boxed{\left[\frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 - \frac{e\hbar}{2m}\sigma.\mathbf{B} - e\Phi\right]\psi = i\hbar\partial_t\psi.} \tag{B.15}$$

This is the Pauli equation in an electromagnetic field. Notice that when the electromagnetic field is removed, the spin operators, σ, vanish and the equation reduces to a form similar to the Schrödinger equation except that the wave function ψ has two components,

$$E\psi - \frac{\mathbf{p}^2}{2m}\psi = 0. \tag{B.16}$$

B.2 Derivation of the Dirac equation

We want to have all space-time derivatives be first order. When we look at the relativistic energy operator, this provides us a clue as to what our new wave equation might look like,

$$E^2 = p^2c^2 + m^2c^4. \tag{B.17}$$

We begin with a possible form of the anticipated first-order operator:

$$\frac{\hbar}{ic}\partial_t = (-\alpha_1 i\hbar\partial_x - \alpha_2 i\hbar\partial_y - \alpha_3 i\hbar\partial_z + \beta mc) \tag{B.18}$$

where α_1, α_2, α_3, and β are undetermined coefficients. Now we square both sides to get:

$$-\frac{\hbar^2}{c^2}\partial_t^2 = (\alpha_1 i\hbar\partial_x + \alpha_2 i\hbar\partial_y + \alpha_3 i\hbar\partial_z - \beta mc)^2 = 0. \tag{B.19}$$

Now we expand the operator on the right-hand side,

$$\begin{aligned} -\frac{\hbar^2}{c^2}\partial_t^2 \quad = \quad & -\alpha_1^2\hbar^2\partial_x^2 - \alpha_1\alpha_2\hbar^2\partial_x\partial_y - \alpha_1\alpha_3\hbar^2\partial_x\partial_z - \alpha_1\beta i\hbar mc\partial_x \\ & -\alpha_2^2\hbar^2\partial_y^2 - \alpha_2\alpha_1\hbar^2\partial_x\partial_y - \alpha_2\alpha_3\hbar^2\partial_y\partial_z - \alpha_2\beta i\hbar mc\partial_y \\ & -\alpha_3^2\hbar^2\partial_z^2 - \alpha_3\alpha_1\hbar^2\partial_x\partial_z - \alpha_3\alpha_2\hbar^2\partial_y\partial_z - \alpha_3\beta i\hbar mc\partial_z \\ & -\beta\alpha_1 i\hbar mc\partial_x - \beta\alpha_2 i\hbar mc\partial_y - \beta\alpha_3 i\hbar mc\partial_z + \beta^2(mc)^2. \end{aligned} \tag{B.20}$$

We can group terms to get

$$\begin{aligned} -\frac{\hbar^2}{c^2}\partial_t^2 \quad = \quad & -\hbar^2(\alpha_1^2\partial_x^2 + \alpha_2^2\partial_y^2 + \alpha_3^2\partial_z^2) + \beta^2(mc)^2 \\ & -\hbar^2[(\alpha_1\alpha_2 + \alpha_2\alpha_1)\partial_x\partial_y + (\alpha_1\alpha_3 + \alpha_3\alpha_1)\partial_x\partial_z + (\alpha_2\alpha_3 + \alpha_3\alpha_2)\partial_y\partial_z \\ & -i\hbar mc[(\alpha_1\beta + \beta\alpha_1)\partial_x + (\alpha_2\beta + \beta\alpha_2)\partial_y + (\alpha_3\beta + \beta\alpha_3)\partial_z]. \end{aligned} \tag{B.21}$$

In order to match the squared energy operator (**??**), we need all first order derivatives to vanish. This places conditions on α_1,α_2,α_3, and β:

$$\alpha_i^2 = 1 \qquad \text{(No summation on } i\text{)} \tag{B.22}$$

$$\beta^2 = 1 \tag{B.23}$$

$$[\alpha_i, \alpha_j]_+ = 0 \tag{B.24}$$

$$[\beta, \alpha_i]_+ = 0 \tag{B.25}$$

where $[\alpha_i, \alpha_j]_+ = \alpha_i\alpha_j + \alpha_j\alpha_i$.

We can see that our original coefficients cannot be simple scalars. We need objects that will anti-commute. Therefore, we shall introduce matrices. We could choose for our representation of these objects the explicit representations given in Appendix **??**. The Dirac equation reads

$$\boxed{\left[(\alpha.\mathbf{p})c + \beta mc^2\right]\Psi = E\Psi.} \tag{B.26}$$

Ψ is now a four-element column matrix.

We can manipulate the Dirac equation further to find an equivalent expression in terms of the γ matrices.

Multiplying by β,

$$\left[\beta(\alpha.\mathbf{p})c + \beta^2 mc^2\right]\Psi = \beta E\Psi, \tag{B.27}$$

and after referring to Appendix **??** for the properties of the β and α matrices, we can rewrite this as

$$\left[(\gamma.\mathbf{p})c + mc^2\right]\Psi = \gamma_0 E\Psi \tag{B.28}$$

$$\left[(\gamma.\mathbf{p}) + mc\right]\Psi = \gamma_0 \frac{E}{c}\Psi \tag{B.29}$$

$$\left(\gamma^i p_i - \gamma_0 \frac{i\hbar}{c}\partial_t\right)\Psi + mc\Psi = 0 \tag{B.30}$$

$$\left(\gamma^\mu p_\mu - mc\right)\Psi = 0 \tag{B.31}$$

where

$$p_\mu = i\hbar\partial_\mu. \tag{B.32}$$

B.2.1 Derivation of spin from the Dirac equation

We need the hamiltonian to commute with the total angular momentum,

$$[\mathbf{H}, \mathbf{J}] = 0, \tag{B.33}$$

where $\mathbf{H} = \alpha.\mathbf{p}c + \beta mc^2$ and $\mathbf{J} = \mathbf{L} + \mathbf{S}$. Working only in the x-component we have

$$\begin{aligned} [\mathbf{H}, \mathrm{L}_x] &= [\alpha.\mathbf{p}c + \beta mc^2, yp_z - zp_y] && \text{(B.34)} \\ [\mathbf{H}, \mathrm{L}_x] &= i\hbar c(\alpha_3 p_y - \alpha_2 p_z) && \text{(B.35)} \end{aligned}$$

which is non-zero. We need an $\mathbf{S}$ such that

$$[\mathbf{H}, \mathrm{S}_x] = -i\hbar c(\alpha_3 p_y - \alpha_2 p_z). \tag{B.36}$$

Suppose we construct the most general angular momentum vector using only the vectors available within the theory:

$$\mathbf{S} = A\alpha + B\beta\alpha + C(\alpha \times \alpha) \tag{B.37}$$

where A, B, and C are scalar constraints.

It follows that

$$\mathrm{S}_x = A\alpha_1 + B\beta\alpha_1 + C(\alpha_2\alpha_3 - \alpha_3\alpha_2). \tag{B.38}$$

Now we evaluate the commutator:

$$[\mathbf{H}, \mathrm{S}_x] = [\alpha_1 p_x c + \alpha_2 p_y c + \alpha_3 p_z c + \beta mc^2\ ,\ A\alpha_1 + B\beta\alpha_1 + C(\alpha_2\alpha_3 - \alpha_3\alpha_2)] \tag{B.39}$$

$$\begin{aligned} [\mathbf{H}, \mathrm{S}_x] = \; & A[\alpha_1, \alpha_1]p_x c + A[\alpha_2, \alpha_1]p_y c + A[\alpha_3, \alpha_1]p_z c + Amc^2[\beta, \alpha_1] \\ & + B[\alpha_1, \beta\alpha_1]p_x c + B[\alpha_2, \beta\alpha_1]p_y c + B[\alpha_3, \beta\alpha_1]p_z c + Bmc^2 \\ & + C[\alpha_1, \alpha_2\alpha_3]p_x c - C[\alpha_1, \alpha_3\alpha_2]p_x c \\ & + C[\alpha_2, \alpha_2\alpha_3]p_y c - C[\alpha_2, \alpha_3\alpha_2]p_y c \\ & + C[\alpha_3, \alpha_2\alpha_3]p_z c - C[\alpha_3, \alpha_3\alpha_2]p_z c \\ & + Cmc^2[\beta, \alpha_2\alpha_3] - Cmc^2[\beta, \alpha_3\alpha_2]. \end{aligned} \tag{B.40}$$

Expanding again we get

$$
\begin{aligned}
[\mathbf{H}, \mathrm{S}_x] = \; & A[\alpha_1,\alpha_1]p_x c + A[\alpha_2,\alpha_1]p_y c + A[\alpha_3,\alpha_1]p_z c \\
& + Amc^2[\beta,\alpha_1] + B[\alpha_1,\beta]\alpha_1 p_x c + B\beta[\alpha_1,\alpha_1]p_x c \\
& + B[\alpha_2,\beta]\alpha_1 p_y c + B\beta[\alpha_2,\alpha_1]p_y c \\
& + B[\alpha_3,\beta]\alpha_1 p_z c + B\beta[\alpha_3,\alpha_1]p_z c + Bmc^2[\beta,\beta]\alpha_1 + Bmc^2\beta[\beta,\alpha_1] \\
& + C[\alpha_1,\alpha_2]\alpha_3 p_x c + C\alpha_2[\alpha_1,\alpha_3]p_x c - C[\alpha_1,\alpha_3]\alpha_2 p_x c - C\alpha_3[\alpha_1,\alpha_2]p_x c \\
& + C[\alpha_2,\alpha_2]\alpha_3 p_y c + C\alpha_2[\alpha_2,\alpha_3]p_y c - C[\alpha_2,\alpha_3]\alpha_2 p_y c - C\alpha_3[\alpha_2,\alpha_2]p_y c \\
& + C[\alpha_3,\alpha_2]\alpha_3 p_z c + C\alpha_2[\alpha_3,\alpha_3]p_z c - C[\alpha_3,\alpha_3]\alpha_2 p_z c - C\alpha_3[\alpha_3,\alpha_2]p_z c \\
& + Cmc^2[\beta,\alpha_2]\alpha_3 + Cmc^2\alpha_2[\beta,\alpha_3] - Cmc^2[\beta,\alpha_3]\alpha_2 - Cmc^2\alpha_3[\beta,\alpha_2].
\end{aligned} \tag{B.41}
$$

By inspection, we can drop terms in which the commutators immediately vanish,

$$
\begin{aligned}
[\mathbf{H}, \mathrm{S}_x] = \; & A[\alpha_2,\alpha_1]p_y c + A[\alpha_3,\alpha_1]p_z c + Amc^2[\beta,\alpha_1] \\
& + B[\alpha_1,\beta]\alpha_1 p_x c + B[\alpha_2,\beta]\alpha_1 p_y c + B\beta[\alpha_2,\alpha_1]p_y c \\
& + B[\alpha_3,\beta]\alpha_1 p_z c + B\beta[\alpha_3,\alpha_1]p_z c + Bmc^2\beta[\beta,\alpha_1] \\
& + C[\alpha_1,\alpha_2]\alpha_3 p_x c + C\alpha_2[\alpha_1,\alpha_3]p_x c - C[\alpha_1,\alpha_3]\alpha_2 p_x c \\
& - C\alpha_3[\alpha_1,\alpha_2]p_x c + C\alpha_2[\alpha_2,\alpha_3]p_y c - C[\alpha_2,\alpha_3]\alpha_2 p_y c \\
& + C[\alpha_3,\alpha_2]\alpha_3 p_z c - C\alpha_3[\alpha_3,\alpha_2]p_z c + Cmc^2[\beta,\alpha_2]\alpha_3 \\
& + Cmc^2\alpha_2[\beta,\alpha_3] - Cmc^2[\beta,\alpha_3]\alpha_2 - Cmc^2\alpha_3[\beta,\alpha_2].
\end{aligned} \tag{B.42}
$$

We expand all the commutators explicitly,

$$
\begin{aligned}
[\mathbf{H}, \mathrm{S}_x] = \; & A(\alpha_2\alpha_1 - \alpha_1\alpha_2)p_y c + A(\alpha_3\alpha_1 - \alpha_1\alpha_3)p_z c \\
& + Amc^2(\beta\alpha_1 - \alpha_1\beta) + B(\alpha_1\beta\alpha_1 - \beta\alpha_1\alpha_1)p_x c \\
& + B(\alpha_2\beta\alpha_1 - \beta\alpha_2\alpha_1)p_y c + B(\beta\alpha_2\alpha_1 - \beta\alpha_1\alpha_2]p_y c \\
& + B(\alpha_3\beta\alpha_1 - \beta\alpha_3\alpha_1)p_z c + B(\beta\alpha_3\alpha_1 - \beta\alpha_1\alpha_3)p_z c + Bmc^2(\beta\beta\alpha_1 - \beta\alpha_1\beta) \\
& C(\alpha_1\alpha_2\alpha_3 - \alpha_2\alpha_1\alpha_3)p_x c + C(\alpha_2\alpha_1\alpha_3 - \alpha_2\alpha_3\alpha_1)p_x c \\
& - C(\alpha_1\alpha_3\alpha_2 - \alpha_3\alpha_1\alpha_2)p_x c - C(\alpha_3\alpha_1\alpha_2 - \alpha_3\alpha_2\alpha_1)p_x c \\
& + C(\alpha_2\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_2)p_y c - C(\alpha_2\alpha_3\alpha_2 - \alpha_3\alpha_2\alpha_2)p_y c \\
& + C(\alpha_3\alpha_2\alpha_3 - \alpha_2\alpha_3\alpha_3)p_z c - C(\alpha_3\alpha_3\alpha_2 - \alpha_3\alpha_2\alpha_3)p_z c \\
& + Cmc^2(\beta\alpha_2\alpha_3 - \alpha_2\beta\alpha_3) + Cmc^2(\alpha_2\beta\alpha_3 - \alpha_2\alpha_3\beta) \\
& - Cmc^2(\beta\alpha_3\alpha_2 - \alpha_3\beta\alpha_2) - Cmc^2(\alpha_3\beta\alpha_2 - \alpha_3\alpha_2\beta),
\end{aligned} \tag{B.43}
$$

and simplify,

$$\begin{aligned}[\mathbf{H}, \mathrm{S}_x] &= -2A\alpha_1\alpha_2 p_y c - 2A\alpha_1\alpha_3 p_z c - 2Amc^2\alpha_1\beta \\ &\quad +2Bmc^2\alpha_1 - 2B\beta p_x c \\ &\quad +4C\alpha_3 p_y c - 4C\alpha_2 p_z c.\end{aligned} \tag{B.44}$$

Remember that we need this to be equal to $-i\hbar c(\alpha_3 p_y - \alpha_2 p_z)$. So now we can compare terms. All the terms are linearly independent. In order to have no mass terms, we must have both $A = 0$ and $B = 0$. This leaves us with only:

$$4C\alpha_3 p_y c - 4C\alpha_2 p_z c = -i\hbar c(\alpha_3 p_y - \alpha_2 p_z). \tag{B.45}$$

Now we can solve for C:

$$C = -\frac{i\hbar}{4}. \tag{B.46}$$

We have determined values for each of the three scalar coefficients A, B, and C. Substituting these values into our original expression for the S_x operator in line (**??**) we find

$$\mathrm{S}_x = -\frac{i\hbar}{4}(\alpha_2\alpha_3 - \alpha_3\alpha_2). \tag{B.47}$$

Now referring to the explicit representations for α given in Appendix **??**, we can write this as

$$\mathrm{S}_x = -\frac{i\hbar}{4}\left(\begin{bmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{bmatrix} - \begin{bmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{bmatrix}\right) \tag{B.48}$$

$$\mathrm{S}_x = -\frac{i\hbar}{4}\left(\begin{bmatrix} \sigma_y\sigma_z & 0 \\ 0 & \sigma_y\sigma_z \end{bmatrix} - \begin{bmatrix} \sigma_z\sigma_y & 0 \\ 0 & \sigma_z\sigma_y \end{bmatrix}\right) \tag{B.49}$$

$$\mathrm{S}_x = -\frac{i\hbar}{4}\left(\begin{bmatrix} i\sigma_x & 0 \\ 0 & i\sigma_x \end{bmatrix} + \begin{bmatrix} i\sigma_x & 0 \\ 0 & i\sigma_x \end{bmatrix}\right) \tag{B.50}$$

$$\mathrm{S}_x = \frac{\hbar}{2}\begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{bmatrix}. \tag{B.51}$$

By symmetry, we will also be able to show that

$$\mathrm{S}_y = \frac{\hbar}{2}\left[\begin{array}{cc} \sigma_y & 0 \\ 0 & \sigma_y \end{array}\right] \tag{B.52}$$

and

$$\mathrm{S}_z = \frac{\hbar}{2}\left[\begin{array}{cc} \sigma_z & 0 \\ 0 & \sigma_z \end{array}\right]. \tag{B.53}$$

If we define a new matrix,

$$\Sigma_i = \left[\begin{array}{cc} \sigma_i & 0 \\ 0 & \sigma_i \end{array}\right], \tag{B.54}$$

we may write our operator **S** as,

$$\boxed{\mathbf{S} = \frac{\hbar}{2}\Sigma,} \tag{B.55}$$

which is the spin operator for particles of spin $\frac{1}{2}$.

B.3 Derivation of the Levy-Leblond equation

We begin first with the Schrödinger equation:

$$i\hbar\partial_t\psi + \frac{\hbar^2}{2m}\nabla^2\psi = 0. \tag{B.56}$$

In this form we recognize the operator,

$$\mathrm{S} \equiv (i\hbar\partial_t + \frac{\hbar^2}{2m}\nabla^2) = \left(E - \frac{\mathbf{p}^2}{2m}\right) = 0, \tag{B.57}$$

where S is the Schrödinger operator. Now following the heuristic approach used by Dirac, we want to find a wave equation which is first-order in all the space and time derivatives. A most general form for this wave equation could be:

$$\Theta\psi \equiv (AE + c\mathbf{B}.\mathbf{p} + mc^2C)\psi = 0 \tag{B.58}$$

in which A, $\mathbf{B}$, and C are linear operators yet to be determined. In order to keep these operators dimensionless, we have introduced extra constants m and c. The introduction of the speed of light, c, bears no connection with special relativity, but is used here simply to ensure the dimensionless nature of the operators, and to allow us to compare with the nonrelativistic limit of the Dirac equation.

For solutions of (**??**) to obey the Schrödinger Equation,

$$\mathrm{S}\psi = 0, \tag{B.59}$$

there must exist some operator

$$\Theta' = (A'E + c\mathbf{B}'.\mathbf{p} + mc^2C') \tag{B.60}$$

such that multiplying (**??**) by Θ' yields the Schrödinger Equation.
In other words we must have

$$\Theta'\Theta = 2mc^2\mathrm{S} \tag{B.61}$$

where the arbitrary coefficient $2mc^2$ provides a convenient normalization.

$$(A'E + c\mathbf{B}'.p + mc^2C')(AE + c\mathbf{B}.p + mc^2C) = 2mc^2\mathrm{S} \tag{B.62}$$

Expanding out the terms in this expression we obtain

$$\begin{aligned} 2mc^2\mathrm{S} \;=\; & (A'A)E^2 + c(A'B_i + B_i'A)p_iE + mc^2(A'C + C'A)E \\ & +m^2c^4C'C + mc^3(C'B_i + B_i'C)p_i \\ & +c^2B_x'B_xp_x^2 + c^2B_y'B_yp_y^2 + c^2B_z'B_zp_z^2 \\ & +c^2(B_x'B_y + B_y'B_x)p_xp_y + c^2(B_x'B_z + B_z'B_x)p_xp_z + \\ & c^2(B_y'B_z + B_z'B_y)p_yp_z. \end{aligned} \tag{B.63}$$

By identifying the various monomials in E and $\mathbf{p}$ and comparing with the energy operator in (**??**), we obtain the following set of conditions on the operators A, **B**, and C:

$$A'A \;=\; 0 \tag{B.64}$$

$$A'B_i + B_i'A = 0 \tag{B.65}$$
$$A'C + C'A = 2 \tag{B.66}$$
$$C'C = 0 \tag{B.67}$$
$$C'B_i + B_i'C = 0 \tag{B.68}$$
$$B_i'B_j + B_j'B_i = -2\delta_{ij}. \tag{B.69}$$

Now, it will help us to find an explicit representation if we define new operators (effectively performing a rotation):

$$B_4 = i(A + \frac{C}{2}) \tag{B.70}$$
$$B_4' = i(A' + \frac{C'}{2}) \tag{B.71}$$
$$B_5 = A - \frac{C}{2} \tag{B.72}$$
$$B_5' = A' - \frac{C'}{2} \tag{B.73}$$

so that

$$\begin{aligned} B_4'B_4 &= -(A' + \frac{C'}{2})(A + \frac{C}{2}) \\ &= -\left[A'A + \frac{1}{2}(A'C + C'A) + \frac{1}{4}C'C\right] \\ &= -1 \end{aligned} \tag{B.74}$$

and

$$\begin{aligned} B_5'B_5 &= (A' - \frac{C'}{2})(A - \frac{C}{2}) \\ &= A'A - \frac{1}{2}(A'C + C'A) + \frac{1}{4}C'C \\ &= -1. \end{aligned} \tag{B.75}$$

Using this rotation we can write a succinct condition on the operators,

$$B_\mu'B_\nu + B_\nu'B_\mu = -2\delta_{\mu\nu}, \tag{B.76}$$

where $\mu, \nu = 1, 2, 3, 4, 5$.

At this point it becomes evident that our B_μ operator must have non-commutative properties and is, therefore, not a scalar. One of the possible representations of these non-commuting objects can be found using the Pauli σ matrices:

$$B_i = \begin{bmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{bmatrix} \qquad B'_i = \begin{bmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{bmatrix} \tag{B.77}$$

$$B_4 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \qquad B'_4 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \tag{B.78}$$

$$B_5 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B'_5 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{B.79}$$

Thus our original coefficients are:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad A' = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{B.80}$$

$$B_i = \begin{bmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{bmatrix} \qquad B'_i = \begin{bmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{bmatrix} \tag{B.81}$$

$$C = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \qquad C' = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \tag{B.82}$$

The reader may check that these matrices satisfy the conditions stipulated in lines (**??**) through (**??**). Since the Pauli matrices are of rank two, the solutions to our wave equation must have four components which we may write as

$$\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{B.83}$$

where ϕ and χ are each two-component spinors.

Substituting our coefficients into our original operator,

$$(AE + c\mathbf{B}.p + mc^2 C)\Psi = 0, \tag{B.84}$$

we finally have

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} + \begin{bmatrix} 0 & -c(\sigma.\mathbf{p}) \\ c(\sigma.\mathbf{p}) & 0 \end{bmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -2mc^2 \end{bmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0 \tag{B.85}$$

$$\begin{bmatrix} E & -c(\sigma.\mathbf{p}) \\ c(\sigma.\mathbf{p}) & -2mc^2 \end{bmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0. \tag{B.86}$$

And our final equation, which we refer to as the Levy-Leblond equation is

$$\boxed{\begin{aligned} -c(\sigma.\mathbf{p})\phi + 2mc^2\chi &= 0 \\ E\phi - c(\sigma.\mathbf{p})\chi &= 0. \end{aligned}} \tag{B.87}$$

If we set the determinant of the matrix in line (**??**) to zero we obtain

$$\mathbf{p}^2c^2 - 2mc^2E = 0 \tag{B.88}$$

$$E - \frac{\mathbf{p}^2}{2m} = 0. \tag{B.89}$$

This is the hamiltonian for nonrelativistic wave equations.

The preceding derivation follows the same arguments presented in Levy-Leblond's original paper. However, Levy-Leblond chooses a slightly different explicit representation for the matrices that satisfy the conditions of (**??**),

$$B'_\mu B_\nu + B'_\nu B_\mu = -2\delta_{\mu\nu}. \tag{B.90}$$

Consequently, Levy-Leblond's final equation, although mathematically equivalent, is of a slightly different form from the equation presented here in line (**??**). Levy-Leblond's actual equation reads

$$\begin{aligned}(\sigma.\mathbf{p})\phi + 2m\chi &= 0 \\ E\phi + (\sigma.\mathbf{p})\chi &= 0\end{aligned}$$

where he has chosen $c = 1$.

Appendix C

Derivation of Probability Currents

C.1 Probability current for the Schrödinger equation in an electromagnetic field

First we define the probability density,

$$\rho = \psi^* \psi, \tag{C.1}$$

and then differentiate the density with respect to time,

$$\partial_t \rho = \psi^* \partial_t \psi + \psi \partial_t \psi^*. \tag{C.2}$$

Now we introduce minimal coupling to the Schrödinger equation for a free particle in an electromagnetic field:

$$(i\hbar\partial_t + e\Phi)\psi = \frac{(\mathbf{p} - e\mathbf{A})^2}{2m}\psi. \tag{C.3}$$

First we expand the operator on the right-hand side,

$$\begin{aligned}
(\mathbf{p} - e\mathbf{A})^2\psi &= (\frac{\hbar}{i}\nabla - e\mathbf{A})(\frac{\hbar}{i}\nabla - e\mathbf{A})\psi \qquad \text{(C.4)} \\
&= -\hbar^2\nabla^2\psi + i\hbar e\mathbf{A}\nabla\psi + i\hbar e\nabla(\mathbf{A}\psi) + e^2A^2\psi \\
&= -\hbar^2\nabla^2\psi + i\hbar e\mathbf{A}\nabla\psi + i\hbar e(\nabla\mathbf{A})\psi + i\hbar e\mathbf{A}\nabla\psi + e^2A^2\psi \\
&= -\hbar^2\nabla^2\psi + 2i\hbar e\mathbf{A}\nabla\psi + i\hbar e(\nabla\mathbf{A})\psi + e^2A^2\psi.
\end{aligned}$$

Substituting this operator back into the Schrödinger equation,

$$i\hbar\partial_t\psi = \frac{1}{2m}\left[-\hbar^2\nabla^2\psi + 2i\hbar e\mathbf{A}\nabla\psi + i\hbar e(\nabla\mathbf{A})\psi + e^2A^2\psi\right] - e\Phi\psi, \quad \text{(C.5)}$$

and multiplying by ψ^* we find

$$\psi^* i\hbar\partial_t\psi = \frac{1}{2m}\psi^*\left[-\hbar^2\nabla^2\psi + 2i\hbar e\mathbf{A}\nabla\psi + i\hbar e(\nabla\mathbf{A})\psi\right] + e^2A^2\psi^*\psi - e\psi^*\Phi\psi. \quad \text{(C.6)}$$

We take the complex conjugate of (**??**),

$$-\psi i\hbar\partial_t\psi^* = \frac{1}{2m}\psi\left[-\hbar^2\nabla^2\psi^* - 2i\hbar e\mathbf{A}\nabla\psi^* - i\hbar e(\nabla\mathbf{A})\psi^*\right] + e^2A^2\psi^*\psi - e\psi^*\Phi\psi. \quad \text{(C.7)}$$

Now subtract equation (**??**) from equation (**??**),

$$i\hbar[\psi^*\partial_t\psi + \psi\partial_t\psi^*] = \frac{1}{2m}\Big[\hbar^2(\psi\nabla^2\psi^* - \psi^*\nabla^2\psi) \quad \text{(C.8)}$$
$$+2i\hbar e\mathbf{A}(\psi^*\nabla\psi + \psi\nabla\psi^*) + 2i\hbar e\psi^*\psi(\nabla\mathbf{A})\Big].$$

We can use (**??**) to substitute on the left-hand side of this result,

$$i\hbar\partial_t\rho = \frac{1}{2m}\Big[\hbar^2(\psi\nabla^2\psi^* - \psi^*\nabla^2\psi) \quad \text{(C.9)}$$
$$+2i\hbar e(\mathbf{A}\psi^*\nabla\psi + \mathbf{A}\psi\nabla\psi^* + \psi^*\psi\nabla\mathbf{A})\Big]$$

$$\partial_t\rho = -\frac{i\hbar}{2m}(\psi\nabla^2\psi^* - \psi^*\nabla^2\psi) + \frac{e}{m}\nabla.(\mathbf{A}\psi^*\psi) \quad \text{(C.10)}$$

$$\partial_t\rho = -\nabla.\left[\frac{i\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi)\right] + \nabla.\left[\frac{e}{m}(\mathbf{A}\psi^*\psi)\right] \quad \text{(C.11)}$$

$$\partial_t\rho + \nabla.\left[\frac{i\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi) - \frac{e}{m}(\mathbf{A}\psi^*\psi)\right] = 0. \quad \text{(C.12)}$$

In this form, we can compare with the continuity equation:

$$\partial_t \rho + \nabla.\mathbf{J} = 0. \tag{C.13}$$

We can read off the probability current:

$$\boxed{\mathbf{J}_{\text{SchrödingerEM}} = \frac{i\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi) - \frac{e}{m}(\mathbf{A}\psi^*\psi).} \tag{C.14}$$

C.2 Probability current for the Pauli equation in an electromagnetic field

We begin with the Pauli equation for an electron in an electromagnetic field:

$$\left[\frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 - \frac{e\hbar}{2m}\sigma.\mathbf{B} - e\Phi\right]\psi = i\hbar\partial_t\psi. \tag{C.15}$$

We make the usual substitution $\mathbf{p} \rightarrow \frac{\hbar}{i}\nabla$,

$$\left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - e\mathbf{A}\right)^2 - \frac{e\hbar}{2m}\sigma.\mathbf{B} - e\Phi\right]\psi = i\hbar\partial_t\psi. \tag{C.16}$$

Just as for the Schrödinger equation, we multiply both sides by $\psi^\dagger$ to get

$$\psi^\dagger\left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - e\mathbf{A}\right)^2 - \frac{e\hbar}{2m}\sigma.\mathbf{B} - e\Phi\right]\psi = i\hbar\psi^\dagger\partial_t\psi \tag{C.17}$$

$$\psi^\dagger\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - e\mathbf{A}\right)^2\psi + \psi^\dagger\left(-\frac{e\hbar}{2m}\sigma.\mathbf{B} - e\Phi\right)\psi = i\hbar\psi^\dagger\partial_t\psi. \tag{C.18}$$

In order to simplify this expression we must expand the operator $(\frac{\hbar}{i}\nabla - e\mathbf{A})^2$. We shall perform this expansion separately:

$$\left(\frac{\hbar}{i}\nabla - e\mathbf{A}\right)^2\psi = -\hbar^2\nabla^2\psi + 2ie\hbar\mathbf{A}(\nabla\psi) + ie\hbar(\nabla\mathbf{A})\psi + e^2\mathbf{A}^2\psi, \tag{C.19}$$

and then substitute into the equation. This yields

$$\psi^\dagger \frac{1}{2m}\Big(-\hbar^2\nabla^2\psi + 2ie\hbar\mathbf{A}(\nabla\psi) + ie\hbar(\nabla\mathbf{A})\psi + e^2\mathbf{A}^2\psi\Big) \qquad \text{(C.20)}$$
$$+\psi^\dagger\left(-\frac{e\hbar}{2m}\sigma.\mathbf{B} - e\Phi\right)\psi \quad = \quad \psi^\dagger i\hbar\partial_t\psi$$

$$-\psi^\dagger\frac{\hbar^2}{2m}\nabla^2\psi + \psi^\dagger\frac{ie\hbar}{m}\mathbf{A}(\nabla\psi) + \psi^\dagger\frac{ie\hbar}{2m}(\nabla\mathbf{A})\psi \qquad \text{(C.21)}$$
$$+\psi^\dagger\left(\frac{e^2}{2m}\mathbf{A}^2 - \frac{e\hbar}{2m}\sigma.\mathbf{B} - e\Phi\right)\psi \quad = \quad \psi^\dagger i\hbar\partial_t\psi.$$

Now take the complex conjugate of equation (**??**), noting that for arbitrary operators **A**, **B**, and **C**,

$$(\mathbf{ABC})^\dagger = \mathbf{C}^\dagger\mathbf{B}^\dagger\mathbf{A}^\dagger, \qquad \text{(C.22)}$$

$$-(\nabla^2\psi^\dagger)\frac{\hbar^2}{2m}\psi - (\nabla\psi^\dagger)\frac{ie\hbar}{m}\mathbf{A}\psi - \psi^\dagger\frac{ie\hbar}{2m}(\nabla\mathbf{A})\psi \qquad \text{(C.23)}$$
$$+\psi^\dagger\left(\frac{e^2}{2m}\mathbf{A}^2 - \frac{e\hbar}{2m}\sigma.\mathbf{B} - e\Phi\right)\psi \quad = \quad -(\partial_t\psi^\dagger)i\hbar\psi.$$

Subtracting equation (**??**) from equation (**??**) leaves

$$\frac{\hbar^2}{2m}\Big[(\nabla^2\psi^\dagger)\psi - \psi^\dagger(\nabla^2\psi)\Big] \qquad \text{(C.24)}$$
$$+\frac{ie\hbar}{m}\Big[\psi^\dagger\mathbf{A}(\nabla\psi) + \psi^\dagger(\nabla\mathbf{A})\psi + (\nabla\psi^\dagger)\mathbf{A}\psi\Big] \quad = \quad i\hbar\Big[\psi^\dagger\partial_t\psi + (\partial_t\psi^\dagger)\psi\Big].$$

If we divide through by $i\hbar$ and group terms, this can be more neatly written as

$$-\frac{i\hbar}{2m}\Big[(\nabla^2\psi^\dagger)\psi - \psi^\dagger(\nabla^2\psi)\Big] + \frac{e}{m}\nabla(\psi^\dagger\mathbf{A}\psi) = \Big[\psi^\dagger\partial_t\psi + (\partial_t\psi^\dagger)\psi\Big]. \quad \text{(C.25)}$$

But we remember that

$$\partial_t \rho = \psi^\dagger \partial_t \psi + (\partial_t \psi^\dagger)\psi, \tag{C.26}$$

so we have

$$-\frac{i\hbar}{2m}\left[(\nabla^2\psi^\dagger)\psi - \psi^\dagger(\nabla^2\psi)\right] + \frac{e}{m}\nabla(\psi^\dagger \mathbf{A}\psi) = \partial_t\rho. \tag{C.27}$$

Again, we have conveniently manipulated the right-hand side to be the exact time derivative of the probability density. Minor rearrangement of terms will allow us to compare with the continuity equation,

$$\partial_t\rho + \nabla.\left[\frac{i\hbar}{2m}\left[(\nabla\psi^\dagger)\psi - \psi^\dagger(\nabla\psi)\right] - \frac{e}{m}(\psi^\dagger \mathbf{A}\psi)\right] = 0. \tag{C.28}$$

Comparing this with the continuity equation,

$$\partial_t\rho + \nabla.\mathbf{J} = 0, \tag{C.29}$$

suggests the probability current to be

$$\boxed{\mathbf{J}_{\text{PauliEM}} = \frac{i\hbar}{2m}\Big[(\nabla\psi^\dagger)\psi - \psi^\dagger(\nabla\psi)\Big] - \frac{e}{m}(\psi^\dagger \mathbf{A}\psi).} \tag{C.30}$$

C.3 Probability current for the Dirac equation

First we define the probability density as

$$\rho \equiv \Psi^\dagger\Psi \tag{C.31}$$

so that

$$\partial_t\rho = (\partial_t\Psi^\dagger)\Psi + \Psi^\dagger(\partial_t\Psi). \tag{C.32}$$

Now we begin with the Dirac equation as found in equation (**??**):

$$\gamma^\mu p_\mu\Psi - mc\Psi = 0, \tag{C.33}$$

where

$$p_\mu = i\hbar\partial_\mu \tag{C.34}$$

$$\partial_0 = \frac{1}{c}\partial_t, \qquad \partial_1 = \partial_x, \qquad \partial_2 = \partial_y, \qquad \partial_3 = \partial_z.$$

Let us remove $\mu = 0$ from the sum in the Dirac equation, and express it explicitly,

$$i\hbar\Big[\gamma^0\frac{1}{c}\partial_t + \gamma^k\partial_k\Big]\Psi - mc\Psi = 0. \tag{C.35}$$

Multiply by $\bar{\Psi}$ where

$$\bar{\Psi} \equiv \Psi^\dagger\gamma^0. \tag{C.36}$$

Also notice that since γ^0 is Hermitian,

$$(\gamma^0)^\dagger = \gamma^0, \tag{C.37}$$

we may write the Hermitian conjugate of $\bar{\Psi}$ as

$$(\bar{\Psi})^\dagger = (\Psi^\dagger\gamma^0)^\dagger = \gamma^0\Psi. \tag{C.38}$$

So, multiplying (**??**) by $\bar{\Psi}$ we get

$$\bar{\Psi}i\hbar\Big[\gamma^0\frac{1}{c}\partial_t + \gamma^k\partial_k\Big]\Psi - \bar{\Psi}mc\Psi = 0 \tag{C.39}$$

$$\frac{i\hbar}{c}\bar{\Psi}\gamma^0(\partial_t\Psi) + i\hbar\bar{\Psi}\gamma^k(\partial_k\Psi) - mc\bar{\Psi}\Psi = 0 \tag{C.40}$$

$$\frac{i\hbar}{c}\Psi^\dagger(\partial_t\Psi) + i\hbar\bar{\Psi}\gamma^k(\partial_k\Psi) - mc\bar{\Psi}\Psi = 0. \tag{C.41}$$

Now we take the complex conjugate of (**??**),

$$-\frac{i\hbar}{c}(\partial_t\Psi^\dagger)\Psi - i\hbar(\partial_k\Psi^\dagger)(\gamma^k)^\dagger\gamma^0\Psi - mc\Psi^\dagger\gamma^0\Psi = 0 \tag{C.42}$$

$$-\frac{i\hbar}{c}(\partial_t\Psi^\dagger)\Psi + i\hbar(\partial_k\Psi^\dagger)\gamma^k\gamma^0\Psi - mc\bar{\Psi}\Psi = 0 \tag{C.43}$$

$$-\frac{i\hbar}{c}(\partial_t\Psi^\dagger)\Psi - i\hbar(\partial_k\bar{\Psi})\gamma^k\Psi - mc\bar{\Psi}\Psi = 0, \tag{C.44}$$

where in line (**??**) we have used the fact that γ^k is antihermitian:

$$(\gamma^k)^\dagger = -\gamma^k. \tag{C.45}$$

Subtracting (**??**) from (**??**),

$$\frac{i\hbar}{c}\Big[\Psi^\dagger(\partial_t\Psi) + (\partial_t\Psi^\dagger)\Psi\Big] + i\hbar\Big[\bar{\Psi}\gamma^k(\partial_k\Psi) + (\partial_k\bar{\Psi})\gamma^k\Psi\Big] = 0, \tag{C.46}$$

and using (**??**) to substitute for the first term on the left,

$$\frac{1}{c}\partial_t\rho + \Big[\bar{\Psi}\gamma^k(\partial_k\Psi) + (\partial_k\bar{\Psi})\gamma^k\Psi\Big] = 0, \tag{C.47}$$

we arrive at

$$\partial_t\rho + \partial_k c(\bar{\Psi}\gamma^k\Psi) = 0. \tag{C.48}$$

Comparing (**??**) with the continuity equation,

$$\partial_t\rho + \partial_k\mathbf{J}^k = 0, \tag{C.49}$$

we may peel off the components of the probability current,

$$\mathbf{J}^k_{\text{Dirac}} = c(\bar{\Psi}\gamma^k\Psi). \tag{C.50}$$

This result may also be expressed as

$$\mathbf{J}^k_{\text{Dirac}} = c(\Psi^\dagger\gamma^0\gamma^k\Psi). \tag{C.51}$$

And then using the fact that

$$\gamma^0\gamma^k = \alpha^k, \tag{C.52}$$

we have

$$\boxed{\mathbf{J}_{\text{Dirac}} = c(\Psi^\dagger\alpha\Psi).} \tag{C.53}$$

C.4 Probability current for the Dirac equation in an electromagnetic field

We begin with the Dirac equation:

$$\gamma^\mu p_\mu \Psi - mc\Psi = 0, \tag{C.54}$$

where

$$p_\mu = i\hbar \partial_\mu \tag{C.55}$$

$$\partial_0 = \frac{1}{c}\partial_t, \qquad \partial_1 = \partial_x, \qquad \partial_2 = \partial_y, \qquad \partial_3 = \partial_z.$$

Now we take this equation into an electromagnetic field using minimal coupling:

$$p_0 \longrightarrow p_0 + \frac{e}{c}\Phi, \qquad \text{and} \qquad p_k \longrightarrow p_k - eA_k.$$

This yields

$$\gamma^0(p_0 + \frac{e}{c}\Phi)\Psi + \gamma^k(p_k - eA_k)\Psi - mc\Psi = 0. \tag{C.56}$$

Now we use the definitions for p_μ given in (**??**):

$$\gamma^0\frac{1}{c}(i\hbar\partial_t + e\Phi)\Psi + \gamma^k(i\hbar\partial_k - eA_k)\Psi - mc\Psi = 0 \tag{C.57}$$

$$\frac{i\hbar}{c}\gamma^0(\partial_t\Psi) + i\hbar\gamma^k(\partial_k\Psi) + \frac{e}{c}\Phi\gamma^0\Psi - e\gamma^k A_k\Psi - mc\Psi = 0. \tag{C.58}$$

We multiply by $\bar{\Psi}$ where

$$\bar{\Psi} \equiv \Psi^\dagger\gamma^0.$$

Also notice that since γ^0 is Hermitian,

$$(\gamma^0)^\dagger = \gamma^0, \tag{C.59}$$

we may write the Hermitian conjugate of $\bar{\Psi}$ as

$$(\bar{\Psi})^\dagger = (\Psi^\dagger\gamma^0)^\dagger = \gamma^0\Psi. \tag{C.60}$$

So, multiplying (**??**) by $\bar{\Psi}$ we get

$$\bar{\Psi}\frac{i\hbar}{c}\gamma^0(\partial_t\Psi) + \bar{\Psi}i\hbar\gamma^k(\partial_k\Psi) + \bar{\Psi}\frac{e}{c}\Phi\gamma^0\Psi - \bar{\Psi}e\gamma^k A_k\Psi - \bar{\Psi}mc\Psi = 0 \quad \text{(C.61)}$$

$$\frac{i\hbar}{c}\Psi^\dagger(\partial_t\Psi) + i\hbar\bar{\Psi}\gamma^k(\partial_k\Psi) + \frac{e}{c}\Phi\Psi^\dagger\Psi - e\bar{\Psi}\gamma^k A_k\Psi - mc\bar{\Psi}\Psi = 0. \quad \text{(C.62)}$$

Now we take the complex conjugate of (**??**) to get

$$\begin{aligned} -\frac{i\hbar}{c}(\partial_t\Psi^\dagger)\Psi - i\hbar(\partial_k\Psi^\dagger)(\gamma^k)^\dagger\gamma^0\Psi + \frac{e}{c}\Phi\Psi^\dagger\Psi & & \quad \text{(C.63)} \\ -e\Psi^\dagger A_k(\gamma^k)^\dagger\gamma^0\Psi - mc\bar{\Psi}\Psi &= 0 & \\ -\frac{i\hbar}{c}(\partial_t\Psi^\dagger)\Psi - i\hbar(\partial_k\bar{\Psi})\gamma^k\Psi + \frac{e}{c}\Phi\Psi^\dagger\Psi & & \quad \text{(C.64)} \\ -e\bar{\Psi}A_k\gamma^k\Psi - mc\bar{\Psi}\Psi &= 0 & \end{aligned}$$

where in line (**??**) we have used the fact that γ^k is antihermitian:

$$(\gamma^k)^\dagger = -\gamma^k. \quad \text{(C.65)}$$

We subtract (**??**) from (**??**),

$$\frac{i\hbar}{c}\Big[\Psi^\dagger(\partial_t\Psi) + (\partial_t\Psi^\dagger)\Psi\Big] + i\hbar\Big[\bar{\Psi}\gamma^k(\partial_k\Psi) + (\partial_k\bar{\Psi})\gamma^k\Psi\Big] = 0. \quad \text{(C.66)}$$

We recognize that the first term on the left contains the time-derivative of the probability density,

$$\frac{1}{c}\partial_t\rho + \Big[\bar{\Psi}\gamma^k(\partial_k\Psi) + (\partial_k\bar{\Psi})\gamma^k\Psi\Big] = 0. \quad \text{(C.67)}$$

We can rewrite this as

$$\partial_t\rho + \partial_k c(\bar{\Psi}\gamma^k\Psi) = 0, \quad \text{(C.68)}$$

and compare with the continuity equation,

$$\partial_t\rho + \partial_k\mathbf{J}^k = 0. \quad \text{(C.69)}$$

From this we may peel off the components of the probability current as

$$\mathbf{J}^k_{\text{DiracEM}} = c(\bar{\Psi}\gamma^k\Psi). \tag{C.70}$$

Notice that the vector potential **A** does not appear explicitly in the final expression for the probability current. The vector potential is however implicitly present in the wave function Ψ, since Ψ is a solution to the Dirac equation where **A** affects the solution.

The probability current may also be expressed as

$$\mathbf{J}^k_{\text{DiracEM}} = c(\Psi^\dagger\gamma^0\gamma^k\Psi) \tag{C.71}$$

And then using the fact that

$$\gamma^0\gamma^k = \alpha^k, \tag{C.72}$$

we have

$$\boxed{\mathbf{J}_{\text{DiracEM}} = c(\Psi^\dagger\alpha\Psi).} \tag{C.73}$$

C.5 Probability current for the Levy-Leblond equation in an electromagnetic field

First we define the probability density,

$$\rho \equiv \phi^\dagger\phi, \tag{C.74}$$

so that

$$\partial_t\rho = \phi^\dagger(\partial_t\phi) + (\partial_t\phi^\dagger)\phi. \tag{C.75}$$

We will need this to solve for **J** in the continuity equation,

$$\partial_t\rho + \nabla.\mathbf{J} = 0. \tag{C.76}$$

Next, we begin with the Levy-Leblond wave equation:

$$\begin{aligned} E\phi - c(\sigma.\mathbf{p})\chi &= 0 \\ -c(\sigma.\mathbf{p})\phi + 2mc^2\chi &= 0. \end{aligned} \tag{C.77}$$

We take this equation into the electromagnetic field using minimal coupling, where $\mathbf{p} \to \mathbf{p} - e\mathbf{A}$, and $E \to E + e\Phi$. With these substitutions, the wave equation becomes

$$\begin{aligned} (E + e\Phi)\phi - c\left[\sigma.(\mathbf{p} - e\mathbf{A})\right]\chi &= 0 \\ -c\left[\sigma.(\mathbf{p} - e\mathbf{A})\right]\phi + 2mc^2\chi &= 0. \end{aligned} \tag{C.78}$$

And now after making canonical substitutions for E and $\mathbf{p}$, we have

$$\begin{aligned} (i\hbar\partial_t + e\Phi)\,\phi - c\left[\sigma.\left(\frac{\hbar}{i}\nabla - e\mathbf{A}\right)\right]\chi &= 0 \\ -c\left[\sigma.\left(\frac{\hbar}{i}\nabla - e\mathbf{A}\right)\right]\phi + 2mc^2\chi &= 0. \end{aligned} \tag{C.79}$$

From the top line of the wave equation we see that

$$\partial_t\phi = -\frac{ic}{\hbar}\left[\sigma.\left(\frac{\hbar}{i}\nabla - e\mathbf{A}\right)\right]\chi + \frac{ie}{\hbar}\Phi\phi \tag{C.80}$$

$$\partial_t\phi = -c\sigma.\nabla\chi + \frac{iec}{\hbar}(\sigma.\mathbf{A})\chi + \frac{ie}{\hbar}\Phi\phi. \tag{C.81}$$

We can use this equation and its adjoint,

$$\partial_t\phi^\dagger = -c\nabla\chi^\dagger.\sigma - \frac{iec}{\hbar}\chi^\dagger(\sigma.\mathbf{A}) - \frac{ie}{\hbar}\phi^\dagger\Phi, \tag{C.82}$$

in our expression for $\partial_t\rho$ in equation (**??**),

$$\begin{aligned} \partial_t\rho &= \phi^\dagger\left[-c\sigma.\nabla\chi + \frac{iec}{\hbar}(\sigma.\mathbf{A})\chi + \frac{ie}{\hbar}\Phi\phi\right] \\ &\quad + \left[-c\nabla\chi^\dagger.\sigma - \frac{iec}{\hbar}\chi^\dagger(\sigma.\mathbf{A}) - \frac{ie}{\hbar}\phi^\dagger\Phi\right]\phi. \end{aligned} \tag{C.83}$$

We can express this result solely in terms of ϕ by using the second line of the wave equation (**??**) to solve for χ,

$$\chi = \frac{1}{2mc}\left[\sigma.(\mathbf{p} - e\mathbf{A})\right]\phi \tag{C.84}$$

$$\chi = -\frac{i\hbar}{2mc}\sigma.\left(\nabla\phi\right) - \frac{e}{2mc}(\sigma.\mathbf{A})\phi. \tag{C.85}$$

Substituting this expression for χ, along with $\chi^\dagger$,

$$\chi^\dagger = \frac{i\hbar}{2mc}(\nabla\phi^\dagger).\sigma - \frac{e}{2mc}\phi^\dagger(\sigma.\mathbf{A}), \tag{C.86}$$

into equation (**??**) gives

$$\begin{aligned}\partial_t\rho &= \phi^\dagger\Bigg[c(\sigma.\nabla)\left[\frac{i\hbar}{2mc}\sigma.(\nabla\phi) + \frac{e}{2mc}(\sigma.\mathbf{A})\phi\right] \\ &\quad -\frac{iec}{\hbar}(\sigma.\mathbf{A})\left[\frac{i\hbar}{2mc}\sigma.(\nabla\phi) + \frac{e}{2mc}(\sigma.\mathbf{A})\phi\right] + \frac{ie}{\hbar}\Phi\phi\Bigg] \\ &\quad +\Bigg[c\nabla\left(-\frac{i\hbar}{2mc}(\nabla\phi^\dagger).\sigma + \frac{e}{2mc}\phi^\dagger(\sigma.\mathbf{A})\right).\sigma \\ &\quad +\frac{iec}{\hbar}\left(-\frac{i\hbar}{2mc}(\nabla\phi^\dagger).\sigma + \frac{e}{2mc}\phi^\dagger(\sigma.\mathbf{A})\right)(\sigma.\mathbf{A}) - \frac{ie}{\hbar}\phi^\dagger\Phi\Bigg]\phi.\end{aligned} \tag{C.87}$$

First notice that the Φ terms cancel, and then after expanding out all the terms we find

$$\begin{aligned}\partial_t\rho &= \frac{i\hbar}{2m}\phi^\dagger(\sigma.\nabla)\sigma.(\nabla\phi) + \frac{e}{2m}\phi^\dagger\left[\sigma.\nabla(\sigma.\mathbf{A}\phi)\right] \\ &\quad +\frac{e}{2m}\phi^\dagger(\sigma.\mathbf{A})\left[\sigma.(\nabla\phi)\right] - \frac{ie^2}{2m\hbar}\phi^\dagger(\sigma.\mathbf{A})^2\phi \\ &\quad -\frac{i\hbar}{2m}\left(\nabla\left[(\nabla\phi^\dagger).\sigma\right].\sigma\right)\phi + \frac{e}{2m}\left(\nabla\left[\phi^\dagger(\sigma.\mathbf{A})\right].\sigma\right)\phi \\ &\quad +\frac{e}{2m}\left[(\nabla\phi^\dagger).\sigma\right](\sigma.\mathbf{A})\phi + \frac{ie^2}{2m\hbar}\phi^\dagger(\sigma.\mathbf{A})^2\phi\end{aligned} \tag{C.88}$$

$$\begin{aligned}\partial_t\rho &= \frac{i\hbar}{2m}\left[\left(\phi^\dagger\sigma_i\sigma_j\nabla_i(\nabla_j\phi) - \nabla_i(\nabla_j\phi^\dagger)\sigma_j\sigma_i\right)\phi\right] \\ &\quad +\frac{e}{2m}\Bigg[\phi^\dagger\left[\sigma_i\nabla_i(\sigma_jA_j\phi)\right] + \phi^\dagger(\sigma_jA_j)\left[\sigma_i(\nabla_i\phi)\right] \\ &\quad +\nabla_i\left[\phi^\dagger(\sigma_jA_j)\right]\sigma_i\phi + (\nabla_i\phi^\dagger)\sigma_i(\sigma_jA_j)\phi\Bigg].\end{aligned} \tag{C.89}$$

In the first line of equation (**??**) we can add and subtract convenient terms so that we can bring one of the derivative operators out on the left side. Also, in the second and third lines, we explicitly evaluate the derivative operators acting on products of functions, and express them as sums of the derivative operator acting on each individual function:

$$
\begin{aligned}
\partial_t \rho \;=\; & \frac{i\hbar}{2m}\Big[\nabla_i\Big(\phi^\dagger\sigma_i\sigma_j(\nabla_j\phi)\Big) - \nabla_i\Big((\nabla_j\phi^\dagger)\sigma_j\sigma_i\phi\Big)\Big] \\
& + \frac{e}{2m}\Big[\phi^\dagger\sigma_i\sigma_j(\nabla_i A_j)\phi + \phi^\dagger\sigma_i\sigma_j A_j(\nabla_i\phi) \\
& + \phi^\dagger\sigma_j\sigma_i A_j(\nabla_i\phi) + (\nabla_i\phi^\dagger)\sigma_j\sigma_i A_j\phi \\
& + \phi^\dagger\sigma_j\sigma_i(\nabla_i A_j)\phi + (\nabla_i\phi^\dagger)\sigma_i\sigma_j A_j\phi\Big]
\end{aligned}
\tag{C.90}
$$

$$
\begin{aligned}
\partial_t \rho \;=\; & \frac{i\hbar}{2m}\nabla_i\Big[\phi^\dagger\sigma_i\sigma_j(\nabla_j\phi) - (\nabla_j\phi^\dagger)\sigma_j\sigma_i\phi\Big] \\
& + \frac{e}{2m}\nabla_i\Big[\phi^\dagger\sigma_i\sigma_j A_j\phi + \phi^\dagger\sigma_j\sigma_i A_j\phi\Big]
\end{aligned}
\tag{C.91}
$$

$$
\begin{aligned}
\partial_t \rho \;=\; & \nabla.\Big[\frac{i\hbar}{2m}\left[\phi^\dagger\sigma\sigma_j(\nabla_j\phi) - (\nabla_j\phi^\dagger)\sigma_j\sigma\phi\right] \\
& + \frac{e}{2m}\left[\phi^\dagger\sigma\sigma_j A_j\phi + \phi^\dagger\sigma_j\sigma A_j\phi)\right]\Big].
\end{aligned}
\tag{C.92}
$$

We compare this with the continuity equation to find

$$
\begin{aligned}
\mathbf{J}_{\mathrm{Levy-LeblondEM}} \;=\; & \frac{i\hbar}{2m}\left[(\nabla_j\phi^\dagger)\sigma_j\sigma\phi - \phi^\dagger\sigma\sigma_j(\nabla_j\phi)\right] \\
& - \frac{e}{2m}\left[\phi^\dagger\sigma\sigma_j A_j\phi + \phi^\dagger\sigma_j\sigma A_j\phi)\right]
\end{aligned}
\tag{C.93}
$$

$$
\begin{aligned}
\mathbf{J}_{\mathrm{Levy-LeblondEM}_i} \;=\; & \frac{i\hbar}{2m}\left[(\nabla_j\phi^\dagger)\sigma_j\sigma_i\phi - \phi^\dagger\sigma_i\sigma_j(\nabla_j\phi)\right] \\
& - \frac{e}{2m}\left[\phi^\dagger\sigma_i\sigma_j A_j\phi + \phi^\dagger\sigma_j\sigma_i A_j\phi)\right]
\end{aligned}
\tag{C.94}
$$

$$\mathbf{J}_{\mathrm{Levy-LeblondEM}_i} = \frac{i\hbar}{2m}\left[(\nabla_j\phi^\dagger)\sigma_j\sigma_i\phi - \phi^\dagger\sigma_i\sigma_j(\nabla_j\phi)\right] - \frac{e}{2m}\left[\phi^\dagger A_j\phi(\sigma_i\sigma_j + \sigma_j\sigma_i)\right]. \quad \text{(C.95)}$$

But,

$$\begin{aligned} \sigma_i\sigma_j &= \delta_{ij}1 + i\epsilon_{ijk}\sigma_k \\ \sigma_j\sigma_i &= \delta_{ji}1 - i\epsilon_{ijk}\sigma_k, \end{aligned} \quad \text{(C.96)}$$

so,

$$\mathbf{J}_{\mathrm{Levy-LeblondEM}_i} = \frac{i\hbar}{2m}\Big[(\nabla_i\phi^\dagger)\phi - \phi^\dagger(\nabla_i\phi) - i\epsilon_{jki}\phi^\dagger(\nabla_j\sigma_k\phi) - i\epsilon_{jki}(\nabla_j\phi^\dagger)\sigma_k\phi\Big] - \frac{e}{2m}\left(2\phi^\dagger A_i\phi\right) \quad \text{(C.97)}$$

$$\mathbf{J}_{\mathrm{Levy-LeblondEM}_i} = \frac{i\hbar}{2m}\left[(\nabla_i\phi^\dagger)\phi - \phi^\dagger(\nabla_i\phi)\right] + \frac{\hbar}{2m}\epsilon_{jki}\nabla_j(\phi^\dagger\sigma_k\phi) - \frac{e}{m}\left(\phi^\dagger A_i\phi\right) \quad \text{(C.98)}$$

$$\mathbf{J}_{\mathrm{Levy-LeblondEM}_i} = \frac{i\hbar}{2m}\left[(\nabla_i\phi^\dagger)\phi - \phi^\dagger(\nabla_i\phi)\right] + \frac{\hbar}{2m}\left[\nabla\times\phi^\dagger\sigma\phi\right]_i - \frac{e}{m}\left(\phi^\dagger A_i\phi\right). \quad \text{(C.99)}$$

Our final result is

$$\boxed{\mathbf{J}_{\mathrm{Levy-LeblondEM}} = \frac{i\hbar}{2m}\left[(\nabla\phi^\dagger)\phi - \phi^\dagger(\nabla\phi)\right] - \frac{e}{m}\mathbf{A}(\phi^\dagger\phi) + \frac{\hbar}{2m}\nabla\times(\phi^\dagger\sigma\phi).} \quad \text{(C.100)}$$

Appendix D

Non-relativistic Limits

D.1 Non-relativistic limits of the Dirac equation

First we write the Dirac equation:

$$\left[(\alpha.\mathbf{p})c + \beta mc^2\right]\Psi = \mathcal{E}\Psi, \tag{D.1}$$

where $\mathcal{E}$ is the total relativistic energy,

$$\mathcal{E} = E + mc^2. \tag{D.2}$$

We begin by expanding the matrix multiplication,

$$\left[\begin{array}{cc} 0 & (\sigma.\mathbf{p})c \\ (\sigma.\mathbf{p})c & 0 \end{array}\right]\left(\begin{array}{c} \psi \\ \chi \end{array}\right) + \left[\begin{array}{cc} mc^2 & 0 \\ 0 & -mc^2 \end{array}\right]\left(\begin{array}{c} \psi \\ \chi \end{array}\right) = \mathcal{E}\left(\begin{array}{c} \psi \\ \chi \end{array}\right) \tag{D.3}$$

$$\begin{array}{rcl} (\sigma.\mathbf{p})c\chi + mc^2\psi & = & \mathcal{E}\psi \\ (\sigma.\mathbf{p})c\psi - mc^2\chi & = & \mathcal{E}\chi \end{array} \tag{D.4}$$

$$\begin{array}{rcl} (\sigma.\mathbf{p})c\chi + mc^2\psi & = & (E + mc^2)\psi \\ (\sigma.\mathbf{p})c\psi - mc^2\chi & = & (E + mc^2)\chi \end{array} \tag{D.5}$$

$$
\begin{aligned}
(\sigma.\mathbf{p})c\chi &= E\psi \\
(\sigma.\mathbf{p})c\psi - mc^2\chi &= (E + mc^2)\chi.
\end{aligned}
\tag{D.6}
$$

In the nonrelativistic limit where $E \ll mc^2$, we can make the approximation

$$
E + mc^2 \approx mc^2. \tag{D.7}
$$

Using this in the second line of the wave equation yields

$$
\begin{aligned}
(\sigma.\mathbf{p})c\chi &= E\psi \\
(\sigma.\mathbf{p})c\psi - mc^2\chi &= mc^2\chi.
\end{aligned}
\tag{D.8}
$$

And now rearranging terms brings us to the Levy-Leblond equation,

$$
\boxed{
\begin{aligned}
-(\sigma.\mathbf{p})\phi + 2m\chi &= 0 \\
E\phi - (\sigma.\mathbf{p})\chi &= 0.
\end{aligned}
}
\tag{D.9}
$$

From this point we require only a few line of calculation to arrive at the Pauli equation. Solving for χ in the bottom line of the Levy-Leblond equation yields

$$
\chi = \frac{(\sigma.\mathbf{p})c}{2mc^2}\psi, \tag{D.10}
$$

which we can substitute into the top line of the Levy-Leblond equation,

$$
E\psi - \frac{(\sigma.\mathbf{p})(\sigma.\mathbf{p})c^2}{2mc^2}\psi = 0 \tag{D.11}
$$

$$
\boxed{E\psi - \frac{\mathbf{p}^2}{2m}\psi = 0.} \tag{D.12}
$$

This is the same form of the Pauli equation as was derived in equation (**??**).

D.2 Nonrelativistic limit of the Dirac probability current in an electromagnetic field

We begin with the Dirac current in an electromagnetic field:

$$\mathbf{J}_{\mathrm{Dirac}} = c(\Psi^{\dagger}\alpha\Psi), \tag{D.13}$$

where Ψ is a four-element column matrix consisting of two two-component spinors ψ and χ,

$$\Psi \equiv \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \tag{D.14}$$

and

$$\alpha_k \equiv \gamma_0\gamma_k. \tag{D.15}$$

We place these definitions in the expression for the Dirac current and perform the matrix multiplication:

$$\mathbf{J}_{\mathrm{Dirac}_k} = c[\begin{pmatrix} \psi^{\dagger} & \chi^{\dagger} \end{pmatrix}\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \chi \end{pmatrix}] \tag{D.16}$$

$$\mathbf{J}_{\mathrm{Dirac}_k} = c[\begin{pmatrix} \psi^{\dagger} & \chi^{\dagger} \end{pmatrix}\begin{pmatrix} \sigma_k\chi \\ \sigma_k\psi \end{pmatrix} \tag{D.17}$$

$$\mathbf{J}_{\mathrm{Dirac}_k} = c[\psi^{\dagger}\sigma_k\chi + \chi^{\dagger}\sigma_k\psi]. \tag{D.18}$$

Equation (**??**) is still a relativistic expression for the Dirac current. We must examine the Dirac equation to see how we shall reduce this expression to a nonrelativistic limit. Following the same methods employed in Section **??** we arrive at the following approximation for χ in the nonrelativistic limit:

$$0 \approx (\sigma.\mathbf{p})\psi - 2mc\chi. \tag{D.19}$$

Introducing minimal coupling, we rewrite this as

$$\chi \approx \frac{1}{2mc}\sigma.(\frac{\hbar}{i}\nabla - e\mathbf{A})\psi. \tag{D.20}$$

We will substitute this into the expression for the probability current in line (**??**),

$$\mathbf{J}_{\mathrm{DiracEMNR}_k} = c\left[\psi^\dagger\sigma_k\left[\frac{1}{2mc}\sigma_i(\frac{\hbar}{i}\nabla_i - eA_i)\psi\right] + \left[\frac{1}{2mc}\sigma_i(\frac{\hbar}{i}\nabla_i - eA_i)\psi\right]^\dagger\sigma_k\psi\right], \tag{D.21}$$

and expand out all the terms,

$$\begin{aligned}\mathbf{J}_{\mathrm{DiracEMNR}_k} &= -\frac{i\hbar}{2m}\psi^\dagger\sigma_k\sigma_i\nabla_i\psi - \frac{e}{2m}\psi^\dagger\sigma_k\sigma_iA_i\psi \\ &\quad +\frac{i\hbar}{2m}(\nabla_i\psi^\dagger)\sigma_i\sigma_k\psi - \frac{e}{2m}\psi^\dagger\sigma_i\sigma_kA_i\psi.\end{aligned} \tag{D.22}$$

Now we remember the properties of the σ matrices from Appendix **??**,

$$\begin{aligned}\sigma_i\sigma_k &= \delta_{ik} + i\epsilon_{ikl}\sigma_l \\ \sigma_k\sigma_i &= \delta_{ki} - i\epsilon_{ikl}\sigma_l,\end{aligned} \tag{D.23}$$

so that our equation becomes

$$\begin{aligned}\mathbf{J}_{\mathrm{DiracEMNR}_k} &= -\frac{i\hbar}{2m}\psi^\dagger(\delta_{ki} - i\epsilon_{ikl}\sigma_l)\nabla_i\psi \\ &\quad -\frac{e}{2m}\psi^\dagger(\delta_{ki} - i\epsilon_{ikl}\sigma_l)A_i\psi \\ &\quad +\frac{i\hbar}{2m}(\nabla_i\psi^\dagger)(\delta_{ik} + i\epsilon_{ikl})\sigma_l\psi - \frac{e}{2m}\psi^\dagger(\delta_{ik} + i\epsilon_{ikl}\sigma_l)A_i\psi.\end{aligned} \tag{D.24}$$

We can expand out each term:

$$\begin{aligned}\mathbf{J}_{\mathrm{DiracEMNR}_k} &= -\frac{i\hbar}{2m}\psi^\dagger\nabla_k\psi - \frac{\hbar}{2m}\psi^\dagger\epsilon_{ikl}\sigma_l\nabla_i\psi \\ &\quad -\frac{e}{2m}\psi^\dagger A_k\psi + \frac{ie}{2m}\psi^\dagger\epsilon_{ikl}\sigma_lA_i\psi + \frac{i\hbar}{2m}(\nabla_k\psi^\dagger)\psi \\ &\quad -\frac{\hbar}{2m}(\nabla_i\psi^\dagger)\epsilon_{ikl}\sigma_l\psi - \frac{e}{2m}\psi^\dagger A_k\psi - \frac{ie}{2m}\psi^\dagger\epsilon_{ikl}\sigma_lA_i\psi,\end{aligned} \tag{D.25}$$

and simplify,

$$\mathbf{J}_{\text{DiracEMNR}_k} = \frac{i\hbar}{2m}\Big((\nabla_k\psi^\dagger)\psi - \psi^\dagger\nabla_k\psi\Big) - \frac{e}{m}\psi^\dagger A_k\psi - \frac{\hbar}{2m}\Big(\psi^\dagger\epsilon_{ikl}\sigma_l\nabla_i\psi + (\nabla_i\psi^\dagger)\epsilon_{ikl}\sigma_l\psi\Big) \tag{D.26}$$

$$\mathbf{J}_{\text{DiracEMNR}_k} = \frac{i\hbar}{2m}\Big((\nabla_k\psi^\dagger)\psi - \psi^\dagger\nabla_k\psi\Big) - \frac{e}{m}\psi^\dagger A_k\psi + \frac{\hbar}{2m}\Big(\psi^\dagger\epsilon_{ilk}\sigma_l\nabla_i\psi + (\nabla_i\psi^\dagger)\epsilon_{ilk}\sigma_l\psi\Big) \tag{D.27}$$

$$\mathbf{J}_{\text{DiracEMNR}_k} = \frac{i\hbar}{2m}\Big((\nabla_k\psi^\dagger)\psi - \psi^\dagger\nabla_k\psi\Big) - \frac{e}{m}\psi^\dagger A_k\psi + \frac{\hbar}{2m}\Big(\psi^\dagger[\nabla\times\sigma\psi]_k + [\nabla\times\psi^\dagger\sigma]_k\psi\Big). \tag{D.28}$$

These three terms form the final result for our nonrelativistic limit of the Dirac current:

$$\boxed{\mathbf{J}_{\text{DiracEMNR}} = \frac{i\hbar}{2m}\Big((\nabla\psi^\dagger)\psi - \psi^\dagger\nabla\psi\Big) - \frac{e}{m}\psi^\dagger\mathbf{A}\psi + \frac{\hbar}{2m}\nabla\times(\psi^\dagger\sigma\psi).} \tag{D.29}$$

Appendix E

Solutions to the Landau Problem

E.1 Solution to the Schrödinger equation for an electron in a homogeneous magnetic field

In order to establish a homogeneous magnetic field, we choose the following vector potential:

$$\mathbf{A} = B_0 x \hat{\mathbf{y}}, \tag{E.1}$$

which gives rise to a magnetic field

$$\mathbf{B} = \nabla \times \mathbf{A} = B_0 \hat{z}. \tag{E.2}$$

Now the Hamiltonian is given by minimal coupling to be

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2, \tag{E.3}$$

so that the time-independent Schrödinger equation becomes

$$\frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 \psi = E\psi. \tag{E.4}$$

Substituting for $\mathbf{p}$ and $\mathbf{A}$ we get

$$\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - eB_0 x \hat{\mathbf{y}}\right)^2 \psi = E\psi. \tag{E.5}$$

Expanding the operator yields

$$\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - eB_0x\hat{\mathbf{y}}\right)\left(\frac{\hbar}{i}\nabla - eB_0x\hat{\mathbf{y}}\right)\psi = E\psi \quad \text{(E.6)}$$

$$\frac{1}{2m}\left[-\hbar^2\nabla^2\psi - \frac{2eB_0\hbar}{i}x\hat{\mathbf{y}}(\nabla\psi) - \frac{eB_0\hbar}{i}(\nabla x\hat{\mathbf{y}})\psi + e^2B_0^2x^2\psi\right] = E\psi \quad \text{(E.7)}$$

$$-\frac{\hbar^2}{2m}\left[\nabla^2\psi + \frac{2eB_0}{i\hbar}x\partial_y\psi - (\frac{eB_0}{\hbar})^2x^2\psi\right] = E\psi. \quad \text{(E.8)}$$

We shall attempt to solve this differential equation using separation of variables. We hope to find an answer that may be expressed in the form

$$\psi = X(x)Y(y)Z(z). \quad \text{(E.9)}$$

With this expression for ψ as our ansatz, the Schrödinger equation (**??**) becomes

$$-\frac{\hbar^2}{2m}\left[X''YZ + XY''Z + XYZ'' + \frac{2eB_0}{i\hbar}xXY'Z - (\frac{eB_0}{\hbar})^2x^2XYZ\right]. = EXYZ \quad \text{(E.10)}$$

Dividing by XYZ and rearranging terms we have

$$-\frac{\hbar^2}{2m}\left[\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} + \frac{2eB_0}{i\hbar}x\frac{Y'}{Y} - \left(\frac{eB_0}{\hbar}\right)^2x^2\right] = E \quad \text{(E.11)}$$

$$-\frac{\hbar^2}{2m}\frac{Z''}{Z} = E + \frac{\hbar^2}{2m}\left[\frac{X''}{X} + \frac{Y''}{Y} + \frac{2eB_0}{i\hbar}x\frac{Y'}{Y} - \left(\frac{eB_0}{\hbar}\right)^2x^2\right]. \quad \text{(E.12)}$$

Clearly, the left-hand side is dependent only upon z, while the right-hand side is dependent only upon x and y. Thus both sides must be equal to a constant. Let us define this constant to be E_z. Then

$$-\frac{\hbar^2}{2m}\frac{Z''}{Z} = E_z \quad \text{(E.13)}$$

$$Z'' + \frac{2mE_z}{\hbar^2}Z = 0. \quad \text{(E.14)}$$

This gives a solution for $Z(z)$,

$$Z(z) = e^{ik_zz}, \quad \text{(E.15)}$$

where

$$k_z \equiv \frac{\pm\sqrt{2mE_z}}{\hbar}. \tag{E.16}$$

Notice here that there is no quantization of the energy in the z-direction. The particle is free to travel in either the positive or negative z-directions.

Now taking the right-hand side of (**??**), and setting it equal to the separation constant E_z we get

$$E_z = E + \frac{\hbar^2}{2m}\left[\frac{X''}{X} + \frac{Y''}{Y} + \frac{2eB_0}{i\hbar}x\frac{Y'}{Y} - \left(\frac{eB_0}{\hbar}\right)^2 x^2\right]. \tag{E.17}$$

Now we let

$$E' = E - E_z, \tag{E.18}$$

so that

$$E' = -\frac{\hbar^2}{2m}\left[\frac{X''}{X} + \frac{Y''}{Y} + \frac{2eB_0}{i\hbar}x\frac{Y'}{Y} - \left(\frac{eB_0}{\hbar}\right)^2 x^2\right]. \tag{E.19}$$

There is no explicit dependence upon y, and thus we may express $Y(y)$ as

$$Y(y) = e^{ik_y y}, \tag{E.20}$$

where k_y is any real number.

Substituting this into (**??**) we find

$$-\frac{\hbar^2}{2m}\left[\frac{X''}{X} - k_y^2 + \frac{2eB_0}{i\hbar}x(ik_y) - \left(\frac{eB_0}{\hbar}\right)^2 x^2\right] = E' \tag{E.21}$$

$$-\frac{\hbar^2}{2m}X'' + \frac{\hbar^2}{2m}\left[k_y^2 - \frac{2eB_0}{\hbar}k_y x + \left(\frac{eB_0}{\hbar}\right)^2 x^2\right]X = E'X \tag{E.22}$$

$$-\frac{\hbar^2}{2m}X'' + \frac{\hbar^2}{2m}\left(\frac{eB_0}{\hbar}x - k_y\right)^2 X = E'X \tag{E.23}$$

$$-\frac{\hbar^2}{2m}X'' + \frac{\hbar^2}{2m}\left(\frac{eB_0}{\hbar}\right)^2\left(x - \frac{\hbar k_y}{eB_0}\right)^2 X = E'X \tag{E.24}$$

$$-\frac{\hbar^2}{2m}X'' + \frac{1}{2}m\left(\frac{eB_0}{m}\right)^2\left(x - \frac{\hbar k_y}{eB_0}\right)^2 X = E'X. \tag{E.25}$$

Now we define the frequency, ω_c, and the x-offset: x_0,

$$\omega_c \equiv \frac{eB_0}{m} \qquad\qquad x_0 \equiv \frac{\hbar k_y}{eB_0} \tag{E.26}$$

This will give

$$-\frac{\hbar^2}{2m}X'' + \frac{1}{2}m\omega_c^2(x-x_0)^2X = E'X. \tag{E.27}$$

This result is in the form Griffiths uses in his book "Introduction to Quantum Mechanics" on page 32 equation [2.39] for the Schrödinger Equation for a simple harmonic oscillator. Using his results on page 40 in equation [2.67] we arrive at the energies:

$$E'_n = (n + \frac{1}{2})\hbar\omega_c. \tag{E.28}$$

Our total energy then is

$$E_{n,k_z} = (n + \frac{1}{2})\hbar\omega_c + E_z \tag{E.29}$$

$$E_{n,k_z} = (n + \frac{1}{2})\hbar\omega_c + \frac{\hbar^2}{2m}k_z^2. \tag{E.30}$$

The energies are infinitely degenerate in k_y, and two-fold degenerate in k_z. The values $\pm k_z$ yield the same energies.

Now to find the resulting wave functions, we solve equation (**??**). The solution is

$$X_n(x) = \left(\frac{m\omega_c}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2} \tag{E.31}$$

where $\xi \equiv \sqrt{\frac{m\omega_c}{\hbar}}(x-x_0)$ and H_n are the Hermite polynomials.

Thus our original wave function, $\psi = X(x)Y(y)Z(z)$, is:

$$\boxed{\psi_{n,k_y,k_z}(x,y,z) = \left(\frac{m\omega_c}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\frac{\xi^2}{2} + ik_y y + ik_z z},} \tag{E.32}$$

where

$$\omega_c = \frac{eB_0}{m} \qquad \xi = \sqrt{\frac{m\omega_c}{\hbar}}(x - x_0) \qquad x_0 = \frac{\hbar k_y}{eB_0}. \tag{E.33}$$

A most general solution is a summation over the three quantum numbers n, k_y, and k_z. Remember that n is an integer, while k_y and k_z are continuous variables.

$$\Psi(x,y,z) = \left(\frac{m\omega_c}{\pi\hbar}\right)^{1/4} \sum_{n=0}^{\infty} \int_{\infty}^{-\infty} dk_y \int_{-\infty}^{\infty} dk_z \quad c_{n,k_y,k_z} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\frac{\xi^2}{2} + ik_y y + ik_z z} \tag{E.34}$$

Now we proceed to analyze the probability current for the electron. We calculate the current vector using the equation

$$\mathbf{J}_{n,k_y,k_z} = -\frac{i\hbar}{2m}[\psi^*_{n,k_y,k_z} \nabla \psi_{n,k_y,k_z} - \nabla \psi^*_{n,k_y,k_z} \psi_{n,k_y,k_z}] \tag{E.35}$$

$$-\frac{e}{m}\psi^*_{n,k_y,k_z} \mathbf{A} \psi_{n,k_y,k_z}. \tag{E.36}$$

The results for the first few energy levels are:

$$\mathbf{J}_{0,ky,kz}(x,y,z) = \sqrt{\frac{m\omega_c}{\pi\hbar}} e^{-\frac{m\omega_c}{\hbar}(x-x_0)^2} \left[0, -\omega_c(x - x_0), \frac{\hbar k_z}{m}\right] \tag{E.37}$$

$$\mathbf{J}_{1,ky,kz}(x,y,z) = \frac{1}{2}\sqrt{\frac{m\omega_c}{\pi\hbar}} \left(2\sqrt{\frac{m\omega_c}{\pi\hbar}}(x - x_0)\right)^2 e^{-\frac{m\omega_c}{\pi\hbar}(x-x_0)^2} \left[0, -\omega_c(x - x_0), \frac{\hbar k_z}{m}\right] \tag{E.38}$$

$$\mathbf{J}_{2,ky,kz}(x,y,z) = \frac{1}{8}\sqrt{\frac{m\omega_c}{\pi\hbar}} \left(4(\frac{m\omega_c}{\hbar})(x - x_0)^2 - 2\right)^2 e^{-\frac{m\omega_c}{\hbar}(x-x_0)^2} \left[0, -\omega_c(x - x_0), \frac{\hbar k_z}{m}\right]. \tag{E.39}$$

Or, for arbitrary n we have

$$\boxed{\mathbf{J}_{n,ky,kz}(x) = \frac{1}{2^n n!}(H_n(\xi))^2 e^{-\xi^2}\left[0, -\omega_c\xi, \sqrt{\frac{\hbar\omega_c}{\pi m}}k_z\right]} \tag{E.40}$$

where

$$\omega_c = \frac{eB_0}{m} \qquad \xi = \sqrt{\frac{m\omega_c}{\hbar}}(x - x_0) \qquad x_0 = \frac{\hbar k_y}{eB_0}. \tag{E.41}$$

Notice that the currents are functions of x only.

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