

REPRESENTATIONS FOR UNDERSTANDING THE
STERN-GERLACH EFFECT

by

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ABSTRACT

REPRESENTATIONS FOR UNDERSTANDING THE STERN-GERLACH EFFECT

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Master of Science

The traditional development of the Stern-Gerlach effect carries with it several very subtle assumptions and approximations. We point out the degree to which this fact has affected the way we practice and interpret modern physics. In order to gain a more complete picture of the Stern-Gerlach effect we apply several techniques and introduce the inhomogeneous Stern-Gerlach effect in which the strong uniform field component is removed. This allows us to identify precession as a critical concept and provide us with a mean to study it in a different and valuable context. Comparison of the various techniques used gives us insight into the applicability of the standard approximations and assumptions. This also provides the context for a more general discussion regarding the use of representations in physics and teaching.

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Express appreciation here.

Contents

Acknowledgments	vi
List of Tables	ix
List of Figures	xi
1 Introduction	1
2 On Representations in Physics	5
2.1 The Necessity of Representations	5
2.2 Mathematical Representations	7
2.2.1 Vectors and Rotations	8
2.2.2 Coordinate Systems	10
2.3 Conceptual Representations	12
2.3.1 Conceptual Systems and Orthogonal Concepts	13
2.3.2 The Early Scientific Revolution	13
2.3.3 Wave-Particle Duality	15
2.4 Representations as Standards	17
3 A Thematic Account of the Stern-Gerlach Effect	19
3.1 The Classical Representation of the Stern-Gerlach Effect	19
3.2 Quantum Mechanics	23
3.2.1 Spin	26
3.2.2 Expectation Values	29
3.2.3 Measurement	29
3.2.4 The Uncertainty Principle	30

3.2.5	Stern and Gerlach's Experiment	31
4	A Historical Account of the Stern-Gerlach Effect	35
4.1	Atomic Models	35
4.1.1	1913: The Bohr Model	36
4.1.2	1916: The Sommerfeld Model	38
4.2	1921-1922: Stern and Gerlach's Experiment	38
4.2.1	Magnet Type	38
4.2.2	Particle Choice	39
4.2.3	Beam Width	39
4.2.4	Results	41
4.3	1925-1926: Quantum Mechanics	41
4.4	1925: Spin	42
4.5	1935: Measurement	43
5	Problems With Accounts of the Stern-Gerlach Effect	45
5.1	Problems from Quantum Mechanics	45
5.1.1	Rightness and Clarity	46
5.1.2	Representations as a Map	47
5.1.3	Communication	47
5.2	Problems with Historical Accounts	48
5.3	Problems with Thematic Accounts	49
5.3.1	An Artificial History	49
5.3.2	Misunderstanding the Practical Aspects of the Stern-Gerlach Effect	51
5.4	The Magnetic Field	52
5.5	Precession Arguments	53
5.5.1	Classical Precession	54
5.5.2	Quantum Precession	55
5.5.3	Precession and the Uncertainty Principle	58

6	The Proposal	61
6.1	Positivist and Realist Representations	62
6.1.1	The Changing Role of Precession	63
6.2	Proposal: The Inhomogeneous Stern-Gerlach Effect	64
6.2.1	The Experimental Arrangement	64
6.2.2	Possible Outcomes	66
7	A More Complete Study of the Stern-Gerlach Effect	67
7.1	Matrix Representation in a Moving Frame	67
7.1.1	The Assumptions	67
7.1.2	The Eigenstates	68
7.1.3	Demonstrating Precession	69
7.1.4	The Inhomogeneous Stern-Gerlach Effect	70
7.2	Position and Momentum Representations	71
7.3	Rotated and Unrotated Representations: Decoupling	73
7.4	Cartesian and Polar Representations: Separation	75
7.5	Series Solution Representation	76
7.5.1	Singularity Structure and Fuch's Theorem	76
7.5.2	The Indicial Equation	77
7.5.3	Recurrence Relation	77
7.5.4	Extracting Behavior	77
7.5.5	Truncation	78
7.5.6	Radius of Convergence	79
7.5.7	Near-Origin Approximation	79
7.6	The Confluent Heun Equation	81
7.7	Clifford Representation	82
7.8	Green's Function Representation	84
7.8.1	A Magnetic Field with Gaussian Fall Off	85
7.8.2	The Born-Approximation	86
7.9	Schrödinger and Heisenberg Representations	87

7.9.1	A Mixed Picture	88
7.9.2	The Heisenberg Picture	90
7.10	Comment on Precession in the Inhomogeneous Stern-Gerlach Effect .	91
7.10.1	The Inhomogeneous Stern-Gerlach Effect as a Local Stern-Gerlach Experiment	91
7.10.2	Making Precession Insignificant: A Semi-Classical Argument .	93
7.11	Other Representations	97
7.11.1	Propagators	97
7.11.2	Numerical Methods	98
7.11.3	Perturbation Theory	98
7.11.4	Bohmian Mechanics	98
8	Conclusions	101
8.1	The Method of Relativity	101
8.2	Our Results	102
8.3	The Dangers of an Inadequate Philosophy	103

List of Tables

2.1	Relating Cartesian and Polar Coordinates	11
2.2	Relating Cartesian and Polar Coordinates	12
2.3	Wave and Particle Concepts	16
5.1	Orthogonal Concepts	46

List of Figures

2.1	The Cartesian Plane	8
2.2	Vector Representations	9
2.3	Function Representations	11
2.4	The Double Slit Experiment and Wave-Particle Duality	14
2.5	A Conceptual Coordinate System	15
3.1	The Classical Model of the Atom	20
3.2	Precession	22
3.3	The Classical Representation of the Stern-Gerlach Effect	23
3.4	Rotations in Physical and “Spin” Space	27
4.1	The Bohr and Sommerfeld Atoms	37
4.2	A Representation of the Stern-Gerlach Apparatus	40
4.3	Beam Cross Section and the Original Stern-Gerlach Results	41
5.1	The Averaging of Components in Precession	55
6.1	The Inhomogeneous Magnetic Field	65
6.2	Four Possible Outcomes for the Inhomogeneous stern-Gerlach Effect	66
7.1	The Radial Separation of the Inhomogeneous Stern-Gerlach Effect	70
7.2	Rotating our Solutions in the Complex Plane	74
7.3	The Parameters for Finding a Constraint on the Observability of the Precessionless Inhomogeneous Stern-Gerlach Effect	94

Chapter 1

Introduction

The early twentieth century was a defining time in physics. In fact, by the mid-1900s a shift of ideas and methods had occurred on such a fundamental level that the discipline of the late 1800s only vaguely resembled the discipline that would close the next century. This shift was fueled by the wrestle between unexpected experimental results and man-made theoretical notions. Amid all this, in 1921 Otto Stern proposed one such experiment that not only validated the monumental shift but helped give it form.

In the Stern-Gerlach experiment a beam of silver atoms was passed through the poles of a magnet. Prior to this time the magnetic field may have been expected to blur the beam into one continuous image due to the magnetic properties of the atoms. However, a few years before it was carried out the idea had been proposed that the magnetic properties of atoms would only manifest themselves in discrete values upon measurement, not continuous ones. In a confirmation of this latest theory the observed trace was indeed quantized. That is, the single beam of atoms did not blur continuously but split into two distinct parts. This was a verification of the emerging doctrine of quantization and, later, of the property of atomic spin.

Because of its simplicity and clarity the Stern-Gerlach experiment has now become axiomatic in modern physics. It is often discussed in textbooks introducing modern ideas and is taken as the clearest demonstration of the quantum measurement process. However, as usually happens with axioms, it *directed* our study to such a large degree that we seldom made it the *object* of our study.

Because it so clearly demonstrates quantization, entanglement, and measurement - all of which are new quantum concepts - the canonization of the usual description of the Stern-Gerlach effect (SGE) has also canonized their usual interpretation.

The usual story of the SGE has gained clout however for good reason. It ties the classical and quantum systems of thought. It seems to explain a clearly quantum result in terms of almost purely classical concepts. For this reason we say it is clear. For this reason it is also approximate.

It was our struggle to understand the classical-sounding story of the SGE in *quantum* terms that led to this thesis. In the textbooks forces, trajectories, precessing vectors were all used to make the description clear while on another page we were forbidden to speak of such things in quantum descriptions (see citeGiffiths for example).

Most of our difficulties seemed to be expressed in the phenomenon of precession so our questions began with comparing its classical and quantum justifications. This led to discussions of a more philosophical nature which in turn led to the interesting discovery that sometimes a problem is too complex to solve singlehandedly. We realized that a single problem could be presented, discussed, and solved in many different ways. By comparing and contrasting these results we found ourselves using a more experimental approach: we solved the same problem in various ways so as to have a sufficient “sample” while only tweeking particular parts and maintaining some “control” variables. After time, this lead to the formulation of our ideas on representations.

Although they appeared last chronologically, we discuss these ideas on the nature and value of representations in the next chapter. In our attempts to make this a general discussion we use non-technical examples as well as technical ones. The former we call Conceptual Representations and use them as examples of our interpretational pictures in physics. The latter we call Mathematical Representations which exemplify several of the solution methods used in later chapters. We relate representations to the paradigms of [2] and discuss the consequence, both good and bad, of canonizing a given representation.

Chapters 3 and 4 use the SGE to show how one phenomenon can be represented in two very different ways. The account of Chapter 3 emphasizes the logical ordering of the concepts that are necessary to understanding the SGE from a quantum perspective. For this reason it is called a Thematic Account and is often used in textbooks [1]. In contrast, we give a Historical Account in Chapter 4. Such an account is characterized by its emphasis of the ordering of concepts and events chronologically. Because the events of history are not always logical this is not as widely found in teaching literature. However, in what follows we attempt to show that such accounts do give an accurate picture of *problem solving* as the attempts of researchers are not always logical either.

In the comparison of the approaches of Chapters 3 and 4 it will be seen how taking either account as *absolute* can lead to problems. Chapter 5 discuss these problems first in terms of the complementary relationship between rightness and clarity. We show how most problems, both technical and philosophical, seem to center on the phenomenon of precession and how the nature of precession depends on the choice of magnetic field.

Having identified the source of most technical and interpretive problems in Chapter 6 we outline a proposal to study the roles of both the magnetic field and precession in the standard description of the SGE. This gives rise to what we call the inhomogeneous SGE (ISGE). The remainder of this work is devoted to understanding it.

In Chapter 7 we attempt to do this. Because of the tension between having a clear account and a right account for the ISGE we choose to compare several different representations and techniques for the problem. Among them are matrix representations (section 7.1), differential calculus techniques (sections 7.2-7.6), Green's functions (section 7.8), and alternative representations for quantum mechanics such as a the Clifford (section 7.7), and Dirac (section 7.9) pictures. In section 7.11 we suggest further approaches that could be used to arrive at a fuller understanding of the ISGE.

In order to place these numerous and varied methods in an appropriate and unifying context we begin with a discussion of the general role of representations.

Chapter 2

On Representations in Physics

Perhaps one of the most subtle aspects of learning to maneuver the physical sciences is gaining an appreciation of and familiarity with representations. They cannot be overly ostentatious or they would obscure the phenomenon of interest. On the other hand, if they are too vague they fail to adequately communicate it. As these two extreme cases are somewhat at odds with each other effectively using various representations is tricky. This difficulty is compounded by our inability to avoid them.

2.1 The Necessity of Representations

In order to express a rational statement it must be presented in a *particular* way. That is, it must be given a representation so that it can be grasped by the mind in terms of a concrete language and based on a set of familiar concepts. For example, when we think of cars we do not have actual cars in our heads. We only represent the concept of cars to our minds as thoughts. So we cannot just think but we must think particular *thoughts*. In other words, because cars are not equivalent to *thoughts* of cars a necessary process of translation takes place. We must therefore constantly represent abstract concepts, either to our mind or to others', in a particular way. The specific manner in which a statement is expressed for its communication or preservation in concrete form constitutes its representation. It is in this sense that representations cannot be avoided.¹

¹For a development of similar ideas see [3] and [4] in which the process of conceptualization is discussed in detail. It is also pointed out that there are many levels of conceptualization, i.e. the perceptual, first order concepts, second order concepts, etc.

An analogous process occurs when translating between two languages. For example, one might say,

(a) *But look how the fish drink in the river.*

or

(b) *Pero mira como beben los peces en el río.*

Using these two statements as different representations of the same idea we can learn several things about representations in general.

(1) Representations are arbitrary *in principle*. Although (a) and (b) are very different expressions they express exactly the same phenomenon.

(2) Representations make assumptions. (a) assumes a knowledge of the English language while (b) assumes a knowledge of Spanish. Because assumptions can be more or less general, representations occur simultaneously on different levels as well. Thus, representations can be “layered” with the more specific ones occurring on top of more general ones.

(3) Representations imply certain things. (a) implies that valid responses are expressible in English whereas (b) implies that Spanish should be used. For spoken languages this may be no surprise but in certain mathematical or physical situations there may be problems or answers that are not easily expressible in a particular way such as explaining the process of electron capture in an ancient African dialect, describing quantum phenomena with only classical concepts, or using a discretely indexed series to express a continuous process.

(4) Representations depend on cultural or philosophic values.² Thus, they are context dependent *interpretations*. To an English speaker (a) is a completely random and detached observation whereas to a Spaniard (b), although describing an identical phenomenon, brings to mind memories and scenes of a religious nature as it is a line from a well known Christmas carol. It is in this sense that the selection and use of a representation or interpretation is a necessary and unscientific part of physics.

²Conversely, cultural and philosophic values often depend on commonly accepted representations and interpretations, e.g. the effects of Newtonian determinism and Darwinian evolution on religious discourse. For this reason, we must be careful about how we represent science in society.

(5) Representations change the communicated meaning of the phenomenon they describe. By utilizing attributes (1)-(4) above representations can be selected to emphasize certain aspects, such as symmetries or biases, to our advantage. Unfortunately, they may also simultaneously and unintentionally obscure other aspects. Making wise choices that properly balance this tradeoff is easily demonstrated but near impossible to teach. This is because this choice is largely an intuitive or instinctive process.³

These are only some of the characteristics of representations as we use the term here. Other will be demonstrated shortly.

2.2 Mathematical Representations

Although there are many types of representations, perhaps as many as there are languages, there are two that are of particular interest in the physics. They might be categorized as mathematical and conceptual representations. The former are typically thought of when representations are mentioned in scientific discourse while the latter also play a very important, though less emphasized, role. We will give a few examples to illustrate these two categories.

One of the earliest examples of a mathematical representation - after students have mastered the ability to accept x as representing an unknown quantity - is the use of the 2-dimensional cartesian grid. By drawing two intersecting lines we gain the ability to express quantitative relationships which we say are 1-dimensional. The choice however of *which* two lines is a choice of representation.

We often make the choice of using an “orthogonal” basis, that is, we choose two lines that not only intersect but that are perpendicular to each other. The fact that they are perpendicular usually provides us with a great simplification over non-orthogonal axes. Also, note that if the information we were trying to represent were more complex we could choose a higher dimensional space, e.g. more perpendicular

³An example of this might be making a judicious choice of coordinates in a Lagrangian problem. This ability seems to be the result of abstracting from experience and not from a concrete or deductive process. As such, this process is not really *taught* but only repeatedly demonstrated by those who have got the “hang of it.” Only some vague guidelines can be given as to how it is done.

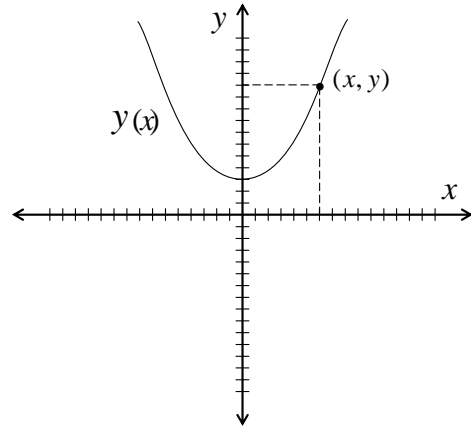


Figure 2.1: On the 2-dimensional Cartesian plane any point can be represented as a couplet of numbers (x, y) specifying its relationship to a predetermined set of coordinate axes. Any set of those points can be written as a function $y(x)$.

axes, by which to represent them. If we were bound to only graphical representations of coordinate systems such as in fig. 2.1 doing this for more than 2 or 3 dimensions would be impossible but because of the more abstract algebraic representations of Descartes' analytic geometry we can easily write functions in n -dimensional spaces as functions of n variables. We will however restrict ourselves to the simplest two-dimensional examples.⁴

2.2.1 Vectors and Rotations

Suppose we have a vector in a 2-dimensional cartesian space. We often represent such an object as a directed line segment. The length of the line segment quantitatively encodes the magnitude and its direction the orientation of the physical quantity in question.

Leaving the vector in this graphical form requires that any mathematical operation involving that vector be done in a graphical manner as well (see attribute (2) above). Thus, we speak of placing vectors, head-to-tail and forming parallelograms, etc. fig. 2.1(a) This is sometimes referred to as the coordinate free *representation*.

⁴Even when we discuss spin system we will restrict ourselves to 2-level systems for reasons discussed in section 3.2.1.

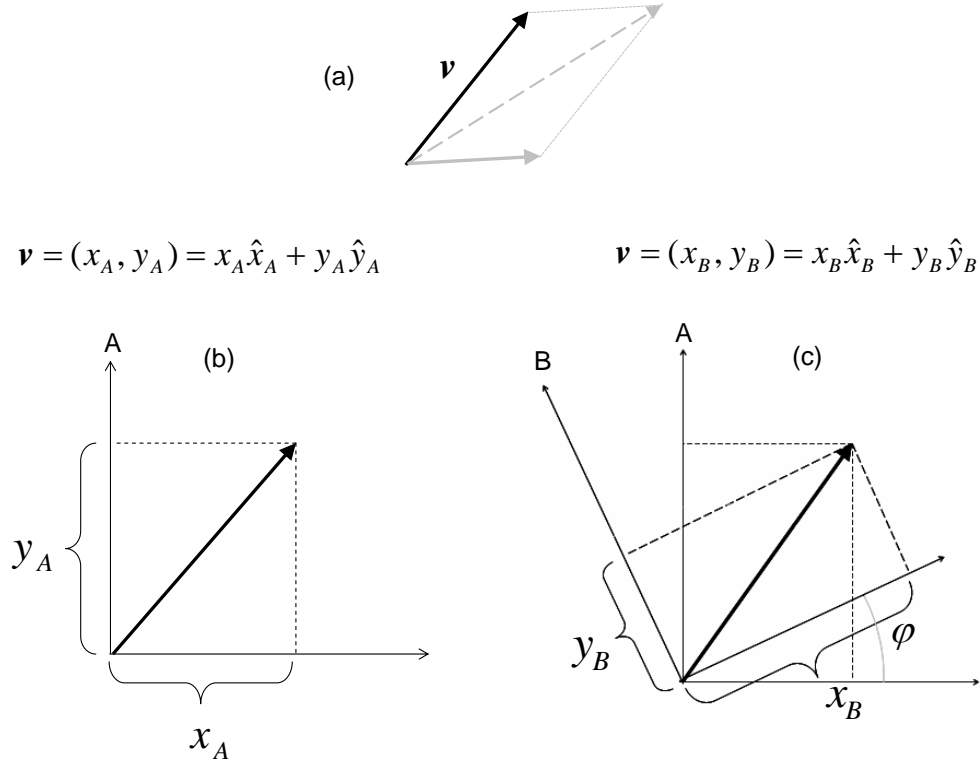


Figure 2.2: (a) Vector v can be represented graphically as a directed line segment. In a graphical manner we can add other vectors to v by forming parallelograms. (b) By imposing an orthogonal coordinate system A we can give our graphical representation a more compact algebraic form $v = x_A \hat{x}_A + y_A \hat{y}_A$. (c) If we rotate A to form a new coordinate system B the *representation* of v has changed to $v = x_B \hat{x}_B + y_B \hat{y}_B$ but the vector itself has not changed at all. Thus, some changes arise from the object itself while others arise only from the representation.

We can however use another layer of representation. If we create a set of perpendicular coordinate axes with which to represent the vector it allows us to use a more compact and general algebraic method to sum or multiply vectors. We are therefore released from the limitations of the graphical methods we mentioned above.

If there were a reason to, we could and often do, arbitrarily rotate the coordinate axes, perhaps to take advantage of a different symmetry. In doing so the length and orientation of the vector are unaffected but its specific mathematical representation in the given basis would change since the axes are now tilted. Although the

components have changed the essential aspects of the vector have not and so any calculation of physical results should be the same regardless of the choice of coordinate system.

If the vector represents a quantity that is horizontal, such as a the displacement of a ball rolling to the right on a table, we often choose to orient the coordinate axes such that the displacement vector lies along the “x”-direction because this direction is typically associated with the horizontal. This choice may reduce the complexity of the problem to that of a 1-dimension (see attribute (5) above).

2.2.2 Coordinate Systems

Another example of a mathematical representation can be given which is of a slightly different sort. Suppose we had two variables with a linear relationship. We could represent it in a cartesian grid by writing

$$y = mx + b, \tag{2.1}$$

which is the familiar equation of a line with slope m and y-intercept b . This form is simple and well known because it takes advantage of the linear symmetry of the given line.

Likewise, if we were asked to represent a circle instead of a line we might have chosen, based on the angular symmetry, a coordinate system that parameterized the angle about the origin. In standard polar coordinates the circle is written as

$$r = a \tag{2.2}$$

where a is the radius. Thus, taking advantage of the known symmetries allows us to express relationships - lines and circles - in very simple ways.

However, if we weren't as experienced with lines and circles or cartesian and polar coordinates we might have chosen, perhaps based on some biased fancy, to express the line in polar and the circle in cartesian coordinates. Although this can be done it disguises the problem in a messy representation. Their algebraic representations become

$$a = \pm\sqrt{x^2 + y^2}, \tag{2.3}$$

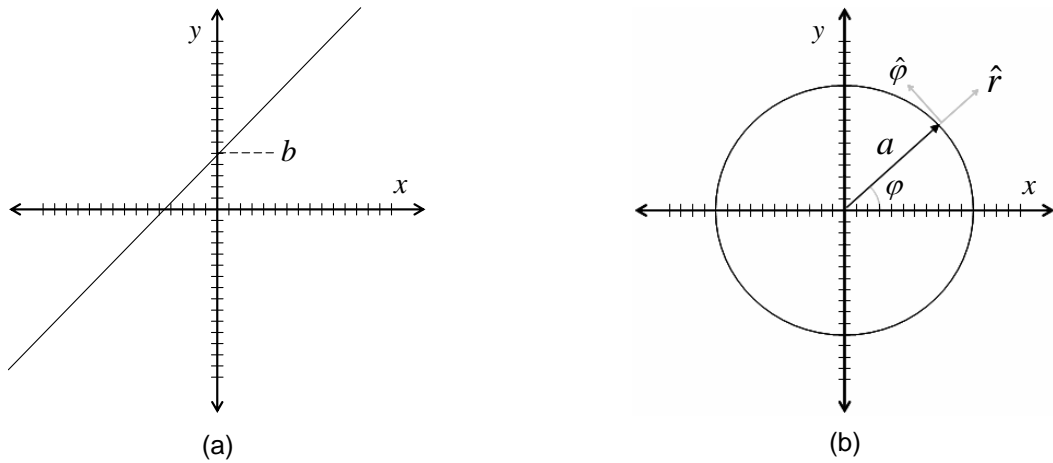


Figure 2.3: (a) Within a coordinate system described by the Cartesian coordinates (x, y) a line is easily described. However, when described by standard polar coordinates (r, ϕ) its simple representation is replaced by a non-linear equation. (b) Conversely, when a circle of radius a is represented it is easy in polar coordinates whereas it is more complicated in Cartesian form.

which is now a piecewise function, for the circle in cartesian coordinates and

$$r = \frac{b}{\sin \varphi - m \cos \varphi}, \quad (2.4)$$

which is now non-linear, for the line in polar coordinates. Many more examples of

	Cartesian Coordinates	Polar Coordinates
line	$y = mx + b$	$r = \frac{b}{\sin \varphi - m \cos \varphi}$
circle	$a = \pm \sqrt{x^2 + y^2}$	$r = a$

Table 2.1: A table comparing Cartesian and Polar coordinate systems.

Translating Between Cartesian and Polar Coordinates	
Cartesian Coordinates \Leftrightarrow Polar Coordinates	
$x = r \cos \varphi$	$r = \pm \sqrt{x^2 + y^2}$
$y = r \sin \varphi$	$\varphi = \arctan\left(\frac{y}{x}\right)$

Table 2.2: A table showing the transformation rules between the Cartesian and Polar coordinate systems.

basis sets⁵, vector spaces, algebras⁶, coordinates systems, and representations⁷ could be given.

These examples not only demonstrate interesting behaviors that properly belong to the representation and not to the curves themselves, i.e. the piecewise nature of eq. (2.3), etc., but also the fact that representations can be layered, that is, you may have representations of representations. This also shows that although the same phenomenon can be given in various ways we often choose among the possibilities in order to emphasize certain aspects. However, if we have no intuitive guide as to what in the problem is worth emphasizing in the problem representations, even though they can still be given, can be unnecessarily messy.

2.3 Conceptual Representations

Just as mathematical representations are utilized only when speaking mathematical languages conceptual representations must be used whenever concepts are used. Their difference is demonstrated in the fact that while mathematics presupposes certain concepts, e.g. numbers, concepts do not obviously presuppose a mathematical language. In this way, conceptual representations can be seen to be more general than

⁵Basis sets can consist of many types of mathematical objects such as vectors, functions, matrices, etc. In fact, even real numbers can be thought of as a 1-dimensional space of which the basis set is the number 1. That is, all numbers can be written as a linear combination of 1's. Basis elements are also usually chosen to be mutually orthogonal and normalized though they need not be.

⁶A specific algebra, the Clifford algebra, will be mention in Chapter 7.

⁷Matrices, for example, require a particular representation.

mathematical ones. If this is the case then it follows that conceptual representations are of extreme importance in physics, for math is. It also follows that selecting a mathematical approach *does* carry conceptual consequences that may affect the approach and results of a problem.⁸ Indeed, specific instances of such representations are used when we employ such things as *models* and *interpretations*, without which science could not progress.⁹

A more suggestive term for all conceptual representations might be *paradigms*. This is meant to allude to [2] which discusses at length the shaping role of paradigms in science. We give shortly two specific examples of paradigm shifts.

2.3.1 Conceptual Systems and Orthogonal Concepts

Before giving some familiar examples of conceptual representations, or paradigms, it is interesting to establish a paradigm of our own to show the similarities in form with the mathematical ones given above. Just as in mathematics great utility is found in representing objects in a coordinate system defined by certain basis elements, conceptual pictures are formed in terms of their own basic set of components. In other words, when a given concept can be *explicitly* identified as a particular weighted combination of a few defining and elementary concepts, much as a vector can be described in terms of its components within a coordinate system, great efficiency and progress can be made. In this light, forming a paradigm is the qualitative equivalent to selecting an appropriate basis in which to describe phenomena. We will use this model in the examples that follow.

2.3.2 The Early Scientific Revolution

A canonical example of a scientific paradigm shift is the early scientific revolution that began with Copernicus and was consummated with Newton. We will not

⁸This addresses a very common opinion that is expressed whenever discussing the research topic of this thesis. Although mathematically “equivalent,” two methods may be markedly different in terms of concepts or pedagogy.

⁹Although many downplay the importance of interpretations in physics today that little progress can be made without them is especially manifest in the concerted efforts that went into formulating a consistent interpretation of quantum mechanics in the first half of the last century. [5]

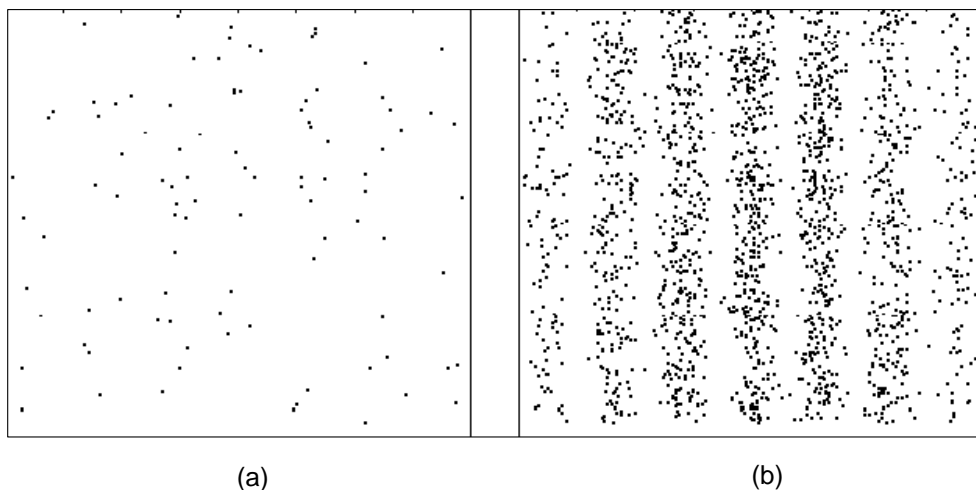


Figure 2.4: (a) If a very low intensity beam of light is sent through a set of narrowly placed slits a pattern of dots accumulates one-by-one on the detecting screen. This is what we'd expect if light were made up of particles. (b) However, if the beam is left for a long time or if the intensity is increased so that several particles strike the scree we can see that the particles (photons) are striking the screen in an orderly fashion, i.e. in a series of bright and dark bands. Because this is exactly what we observe with all wave phenomena, such as with sound or water waves, this is what we'd expect from if light were a wave. Therefore, we see that while the path of light is wave-like (as in (b)) the way in which it collides with detectors is particle-like (as in (a)).

go into the historical details but we can see that the change that occurred during this time was primarily one of perspective. The gods had not altered their course, neither had the planets changed their motion, and yet physics was revolutionized and completely changed form. Simply put, before the revolution we operated in a conceptual “space” - a conceptual “coordinate” system - that set the concepts of simplicity, anthropocentrism, and the duality of the terrestrial and divine natures among others as the basis whereas after Copernicus, Kepler, Galileo, and Newton had done their work we saw different advantages and accordingly shifted our values to one that enshrined objectivity, mechanism, reductionism, determinism, and mathematical rigor. Our way of conceptualizing - of seeing problems - had been drastically altered. We had effectively rotated or redefined our conceptual system so as to take advantage of our new found values much as we did in section 2.1. As a result the way in which we

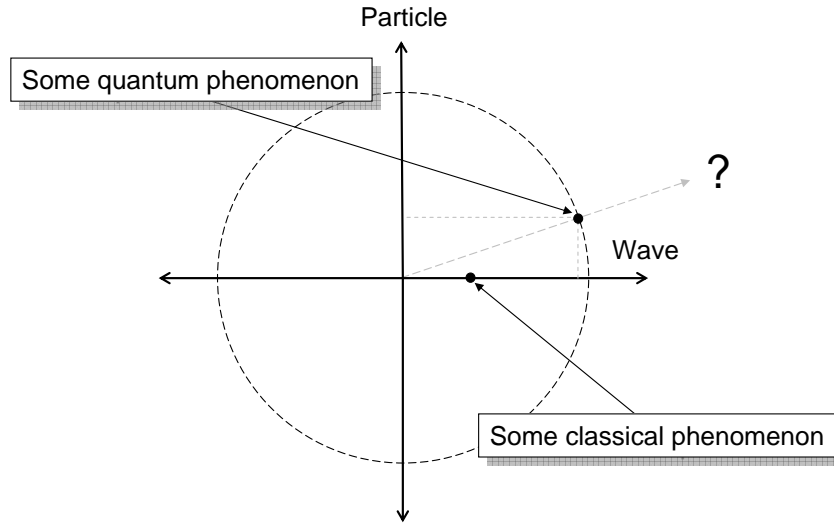


Figure 2.5: Because the duality demonstrated in the 2 slit experiment of fig. 2.4 is so common in fully describing quantum phenomena we have realized the necessity of using a 2-dimensional conceptual system. Each of the basis elements are classical concepts so only a 1-dimensional system is needed to describe a phenomenon classically. We will see in section 3.2.4 that any *representation* in terms of these dual, or complementary, concepts is constrained to something like a circle, i.e. as a system is more describable in terms of particles it is correspondingly less describable in terms of waves. The question remains as to whether the phenomena *themselves* can evolve *off* the circle.

interpreted and categorized our problems, methods, results, and even values changed as well.¹⁰

2.3.3 Wave-Particle Duality

There are more modern examples of paradigm shifts. One that has not resulted in the final selection of one system completely at the expense of another, as did the early scientific revolution, is couched in quantum mechanics.¹¹ Here we have come to grips with the need to adopt at least two very different paradigms. We have learned to constantly shift our view between the two based on the nature of the questions asked. This is because, in the language of mathematical spaces, our concepts have been reduced to not just any set of concepts but to what we might call an *orthogonal*

¹⁰For an account of this period and an idea of the concepts that formed the conceptual “space” in which these theories developed see [6] or [7].

¹¹See [8] for an excellent discussion of the history and development of many oddities in quantum mechanics including wave-particle duality.

set of concepts. Bohr called these sorts of concepts “complementary” and developed many ideas based on the principle of complementarity. [19] Similar to the definition given in section 2.1 by orthogonal concepts we mean two concepts that are in no way expressible in terms of each other. If we take light, for example, as our object of description these two concepts are waves and particles.

Particle Concepts	Wave Concepts
Mass	Frequency
Trajectory	Amplitude
Collisions	Phase
Position	Wave Fronts
Force	Peaks

Table 2.3: A table of some concepts that we use to describe wave and particles. Particle concepts are not well defined when applied to waves and visa versa. The set of concepts listed in either column separately form a conceptual space or basis, much as the set $(\hat{x}, \hat{y}, \hat{z})$ does, within which the overall concept of “particle” or “wave” can be clearly used.

Some experiments show light to demonstrate the properties of waves while others demonstrate the properties of particles. Because there are at least these two orthogonal ways of representing light there are at least two completely different sets of properties of which we can learn. As a particle, we can use the well-defined concepts of position, speed, mass, trajectory, and force as applied to light while as a wave we may use wavefronts, interference, crests, troughs, frequency, and amplitude. It seems that depending on the precise experimental questions we ask we may use either the attributes and concepts of waves or particles in studying light.

It is interesting to note that we have effectively doubled the space in which we describe quantum mechanics by doubling the the number of concepts that can be applied to it. This is the reverse of the vector example above in which we chose a mathematical representation so as to reduce the dimensionality of the problem. Here

we choose a conceptual representation that *increases* the conceptual dimensionality of the problem in order to give it a more complete representation; the problem can now be described as some combination of “waviness” and “particle-ness” just as a vector might be describable in its x and y orthogonal components. This is not only analogous to the introduction of spin into quantum mechanics into which the spin degree of freedom was added in order to account for observed phenomena but it is gist of the entire approach of this work. We attempt to expand the conceptual space available to describing spin phenomena as manifest in the Stern-Gerlach effect. This demonstrates the practical effects a shift in conceptual representation may have.

2.4 Representations as Standards

When the utility of a particular representation is demonstrated with respect to a shared value system, whether mathematical or conceptual, they can become standard, or axiomatic. As such, new hypotheses and results are compared against it in order to gauge their validity and “truthfulness.” If there is an inconsistency it is the nature of the hypothesis or result that is usually questioned and not the standard.¹² This can have several undesirable consequences some of which are

(1) Their democratically-decided status as “standard” implies to many minds the absolute status of *unquestionable*.

(2) Once considered standard the necessity and prevalence of representations can easily be mistaken for complete objectivity.

(3) The characteristics of nature that they assume are seen as naturally or mathematically imposed.

(4) The implications and emphases of the representation become confused with those of the original phenomena.

(5) The solution methods and interpretations natural to the representation are seen as required.

¹²When enough discrepancies arise there may come a point when the standards themselves begin to be questioned. This occurs at the onset of scientific revolutions. [2]

(6) The value system used to determine it and that arises from it are mistaken as absolute.

In short, all unscientific aspects of the representation process can easily and mistakenly be given scientific status. When they thus become axiomatic representations they are then often taken for granted and much valuable insight is lost.

On the other hand, standards and established norms - even to the hiding of irregularities and messiness - are crucial to the communication and development of any science. Therefore, sifting these subtleties out scientifically, not necessarily abandoning them, is necessary for further progress.

Chapter 3

A Thematic Account of the Stern-Gerlach Effect

Representations come in many forms. They may be qualitatively expressed in everyday language or quantitatively expressed in mathematical form. In any case, a particular representation is chosen in order to emphasize a particular aspect of the phenomenon being described. In this chapter we choose to represent the Stern-Gerlach effect (SGE) in a manner that will appear very similar to those given in various textbooks (see [1] or [9]). We will seek to emphasize the logical ordering of the themes and concepts of the SGE in order to provide a clear and rational account. For this reason we call it a thematic account. In the next chapter the same story will be told but from a historical perspective.

3.1 The Classical Representation of the Stern-Gerlach Effect

We begin with a classical description of the SGE because it introduces the main concepts with which students are usually familiar when it is first presented. It therefore provides an appropriate context for the quantum mechanical “textbook” description we subsequently give.

In the classical picture of the atom we treat the electron as tracing a definite orbit around the nucleus. This moving charge creates a current I that encloses a vector area \mathbf{A} , with direction \mathbf{n} normal to the surface in a righthanded sense with the orbital motion. We may thus use the familiar equation for magnetic moments $\boldsymbol{\mu}$ from classical electromagnetic theory

$$\boldsymbol{\mu} = I\mathbf{A}. \tag{3.1}$$

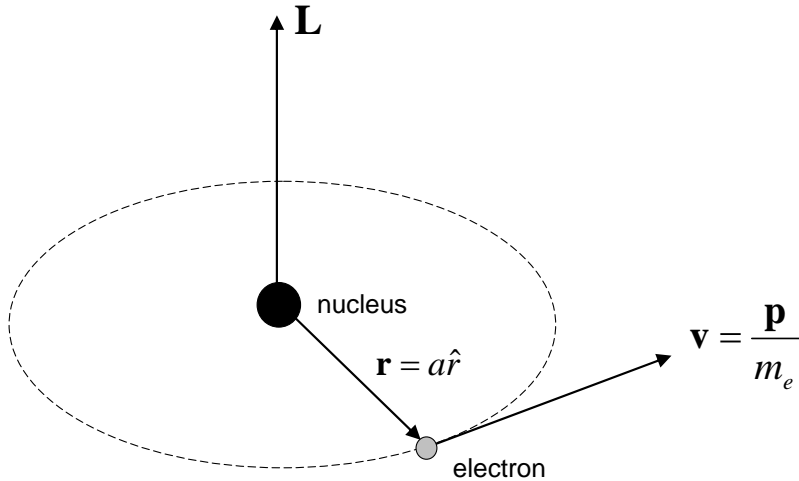


Figure 3.1: In the classical picture of the atom the electron orbits the nucleus much as a planet would orbit the sun. Because it has a continuous set of definite trajectories with velocity \mathbf{v} and momentum $\mathbf{p} = m_e \mathbf{v}$ there is a definite and continuous orbital angular momentum $\mathbf{L} = m \mathbf{r} \times \mathbf{v}$.

More specifically, if we assume that charge e orbits κ times¹ around a circular path of radius a with velocity \mathbf{v} such that the period T of the particle is $T = 2\pi a/v$ we may write $I = \kappa e/T = \kappa e v/2\pi a$. Using $\mathbf{A} = \pi a^2 \mathbf{n}$ we have

$$\boldsymbol{\mu} = \frac{\kappa e a v \mathbf{n}}{2}. \quad (3.2)$$

Multiplying and dividing by the mass m of the particle allows us to recognize the total angular momentum $\mathbf{L} = a m v \mathbf{n}$. We now have a general magnetic moment

$$\boldsymbol{\mu} = \frac{\kappa e}{2m} \mathbf{L} \quad (3.3)$$

for electrons. In the classical picture we assume that the particle carries the charge so, since in one period T the particle makes one complete revolution, so does the charge. Classically we take $\kappa = 1$. The reason for introducing κ will become more apparent in section 3.2.1.

The energy of $\boldsymbol{\mu}$ in a magnetic field \mathbf{B} is

$$U = -\boldsymbol{\mu} \cdot \mathbf{B} \quad (3.4)$$

¹Usually $\boldsymbol{\mu}$ is derived classically without any mention of κ , since it is 1. We use this approach here, which is not at all standard, to show two things. However we leave this discussion to section 3.2.1.

and the dynamics are defined by

$$\mathbf{F} = -\nabla U = \nabla(\boldsymbol{\mu} \cdot \mathbf{B}) \text{ and } \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \boldsymbol{\mu} \times \mathbf{B}. \quad (3.5)$$

Using eq. (3.3) inverted for \mathbf{L} and defining a characteristic frequency $\boldsymbol{\omega} \equiv -\kappa e\mathbf{B}/2m$ the second of these equations becomes

$$\frac{d\boldsymbol{\mu}}{dt} = \boldsymbol{\omega} \times \boldsymbol{\mu}. \quad (3.6)$$

Thus, as can be seen in fig. 3.2, the particle's magnetic moment precesses about \mathbf{B} at a frequency ω .

As this aspect of the dynamics evolves the force equation also has an effect. Under the very common though very subtle assumption that $\boldsymbol{\mu}$ does not vary in space the gradient operator in \mathbf{F} of eq. (3.5) moves past $\boldsymbol{\mu}$ giving the components, in index notation, $F_j = \mu_k \partial_j B_k$. Therefore, the magnitude of the force felt by the particle will be proportional to both the magnitude of $\boldsymbol{\mu}$ and of $\nabla\mathbf{B}$ as well as orientation of $\boldsymbol{\mu}$ relative to the field. The direction of \mathbf{F} arises from the direction of the field gradient. This means that the particles will sift themselves out according to the magnitude and direction of their respective magnetic moments as they precess. The *magnitude* of the deflection depending on the magnitude of $\boldsymbol{\mu}$ in the direction of the field gradient and the *direction* of the deflection being *only* a function of $\boldsymbol{\mu}$'s *local* direction. Said differently, the *fact* of deflection reflects only the orientation of $\boldsymbol{\mu}$ with respect to a chosen field configuration while the *direction* of deflection communicates information about the particle's position in the field.

In order to classically derive the SGE we imagine passing a beam of particles each with a random magnetic moment $\boldsymbol{\mu}$ through a magnetic field that incorporates both characteristics of interest: a field gradient b that induces deflections and a uniform part that directs them a certain way. The simplest choice is

$$\mathbf{B} = (B_0 + bz)\hat{z}. \quad (3.7)$$

Applying this to eq. (3.5) yields

$$\mathbf{F} = \mu b \hat{z} \quad (3.8)$$

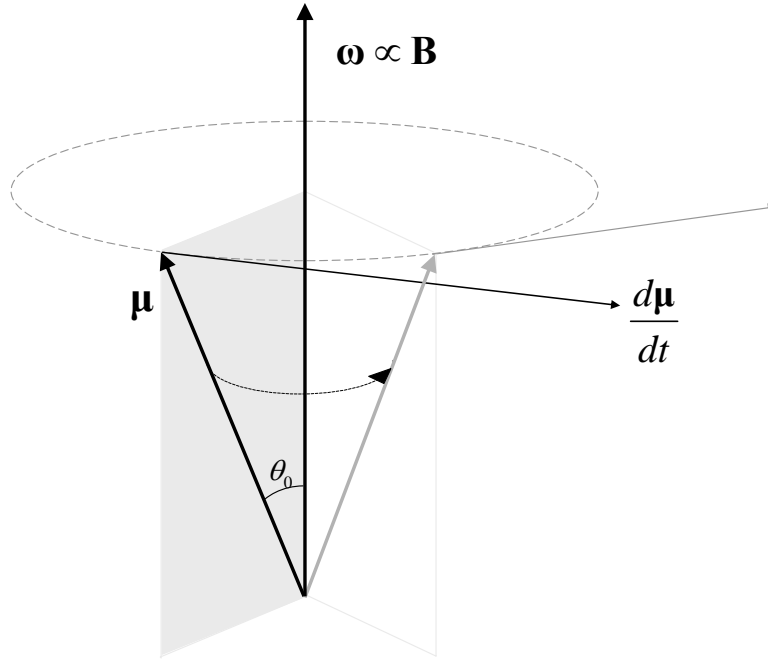


Figure 3.2: The vector $\boldsymbol{\mu}$ will move in a direction that is perpendicular to both itself and the magnetic field direction \mathbf{B} . As it thus moves its orientation changes. At every point $\boldsymbol{\mu}$ will tend to move in a direction tangent to the circle centered on and perpendicular to \mathbf{B} . Thus, $\boldsymbol{\mu}$ rotates or precesses about \mathbf{B} at a frequency of ω . Note that if there were some dissipative force in this system $\boldsymbol{\mu}$ would align with \mathbf{B} as it precessed. This is what happens to a compass needle because of the friction of the pivot point.

as the equation of motion. Thus, in the classical representation of the experiment we expect each particle to move under a force that is proportional to $\boldsymbol{\mu}$ and the field gradient b in the direction of $\nabla\mathbf{B}$. Inasmuch as the orientation of $\boldsymbol{\mu}$ is continuous and $\nabla\mathbf{B}$ is unidirectional a continuous, linear distribution of particles will register on a detecting plate placed some distance behind the magnet. This is the classical description of the SGE.

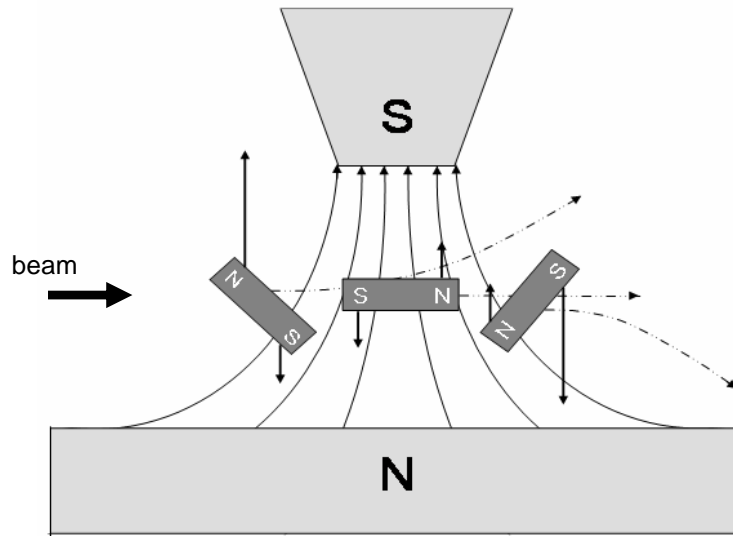


Figure 3.3: A beam of atoms, whose magnetic properties can be represented as tiny bar magnets, enters an inhomogeneous magnetic field from the left. Depending upon the orientation of the bar magnetic in the field a net force will be exerted. This force causes a corresponding separation of the trajectories.

3.2 Quantum Mechanics

The quantum description of this experiment is much different. Before going into its derivation specifically a thematic approach requires that we introduce some concepts that will be needed but that are not part of the classical intuition we have built up. Together with the classical concepts, these will form a common conceptual context from which students seeing the SGE for the first time will draw.

The governing equation of quantum mechanics is Schrödinger's equation. In general form it is

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle \quad (3.9)$$

where \hat{H} is the Hamiltonian operator and $|\Psi\rangle$ represents a general quantum state.²

In light of the previous chapter we note that by “general” we mean here only that it is free of *particular kinds* of representations not of *any* kind of representation. That is, $|\Psi\rangle$ is a general state in Hilbert space but has not yet been expressed in terms of a more narrow space, such as position or momentum space. Once we move to one such space this will be analogous to the example in section 2.2.1 when a vector’s geometric, coordinate free representation as a directed line segment was made more concrete and useful through the introduction of a particular basis.

Because \hat{H} is the Hamiltonian operator we may write it in a more suggestive form

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}, \quad (3.10)$$

where we now have the square of the momentum operator $\hat{\mathbf{p}}^2$ and the potential energy operator \hat{V} .

We can represent eq. (3.9) using eq. (3.10) in a familiar way by projecting the states and operators into a coordinate basis defined by the triplet (x, y, z) . We will call this \mathbf{x} -space.³ We also realize that $|\Psi\rangle$ may carry time-dependence so $|\Psi\rangle \rightarrow \Psi(t)$. This gives us the Schrödinger representation which is most commonly dealt with in introductory treatments. With these choices⁴ we follow the prescribed formalism of quantum mechanics for projecting into \mathbf{x} -space and multiply everything on the left with $\langle \mathbf{x}|$. We get,

$$\langle \mathbf{x}|\Psi(t)\rangle = \Psi(\mathbf{x}, t) \quad (3.11)$$

$$\langle \mathbf{x}|\hat{p}^2\Psi(t)\rangle = -\hbar^2\nabla^2\Psi(\mathbf{x}, t) \quad (3.12)$$

$$\langle \mathbf{x}|\hat{V}\Psi(t)\rangle = V(\mathbf{x})\Psi(\mathbf{x}, t). \quad (3.13)$$

At this point we can see that it will be economic for us to suppress the explicit listing of functional dependencies when it is convenient to do so. They will still be included in the most general equations or when the meaning is unclear without them.

²We use $\hat{}$ to differentiate operators whether matrix, vector, or scalar from other mathematical objects except in the case of ∇ for which we think it is obvious. Unfortunately the $\hat{}$ is also standard notation for unit vectors. When the context doesn’t make this clear we’ll make it more explicit.

³We will work in \mathbf{p} -space in chapter 6.

⁴Later we will demonstrate the usefulness of other choices of representation.

Inserting these specified forms into eq. (3.9) we get

$$i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V\Psi. \quad (3.14)$$

This is the time-dependent Schrödinger equation represented in \mathbf{x} -space.

Because this is now represented as a familiar differential equation we can use familiar differential solution techniques to proceed. If \hat{H} is time-independent, which implies that \hat{V} is as well, we can use the separation of variables technique to separate the time behavior out. Assuming

$$\Psi(\mathbf{x}, t) = \psi(\mathbf{x})T(t) \quad (3.15)$$

and calling the constant of separation E yields the two equations

$$ET = i\hbar \frac{d}{dt} T \quad (3.16)$$

$$E\psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + \hat{V}\psi \quad (3.17)$$

for the time and space parts respectively. Note that the spatial equation is an eigenvalue equation of the form

$$\hat{H}\psi_n = E_n\psi_n \quad (3.18)$$

with eigenvalues E_n corresponding to eigenfunctions ψ_n and also that the partial time derivative in eq. (3.16) can become a total derivative because of the lack of spatial dependence in T . We will suppress the index n for now.

Solving the time piece gives

$$T(t) = T(0)e^{-\frac{i}{\hbar}Et}. \quad (3.19)$$

A general solution of the full equation is then the linear combination

$$\Psi(\mathbf{x}, t) = \sum_{n=0}^N \psi_n(\mathbf{x})e^{-\frac{i}{\hbar}E_n t} \quad (3.20)$$

where N , the dimensionality of the quantum space, could be either finite or infinite. All constants have been absorbed into $\psi_n(\mathbf{x})$.

Had we represented our operators as N -dimensional matrices instead of algebraic operators we could have arrived at the same result. This would be a useful choice for systems of only a few dimensions such as spin systems.

3.2.1 Spin

Spin is a necessary form of angular momentum in the quantum description of nature. It is analogous to the classical orbital angular momentum $\hat{\mathbf{L}}$ although it has not yet been consistently associated with a conceptual picture of any spinning or orbiting object. Orthodox interpretations of quantum mechanics take it to have no classical manifestation and to be intrinsic to quantum objects such as photons and electrons.

As a measure of a quantum angular momentum spin $\hat{\mathbf{S}}$ satisfies the same equations as $\hat{\mathbf{L}}$. In particular,

$$\hat{\boldsymbol{\mu}} = \frac{\kappa e}{2m} \hat{\mathbf{S}} \quad (3.21)$$

corresponding with eq. (3.3) but with an important difference. For quantum mechanical descriptions of electrons it has been found that there is a missing factor of 2. There are several ways to introduce this correction. One possibility is to introduce a new factor $g = 2$ that didn't have an appreciable effect in the classical description and whose interpretation is consequently unclear. Another is to assume that κ changes values in moving from the classical to the quantum description. If this is the case, then our picture also changes. Based on the interpretation we gave with eq. (3.3) with $\kappa = 2$ instead of 1 the charge apparently rotates *twice* for every *single* revolution of the particle. Although there is no mathematical difference, only interpretive turmoil, let us rename $\kappa \rightarrow g$.⁵ If this is the case then for one full revolution of the particle in

⁵As mentioned in foot note 1 we do all this for two reasons. First, this shows how easily one can attach a meaning to a symbol because of the way it is used or written. If we have initially called κ g those familiar with the Landé factor of quantum mechanics would have immediately associated the two. However, if one has never used the letter g in this context, as most students of the classical picture haven't, that wouldn't be a concern. To create this effect and to show how easily mental pictures can affect mathematical details we chose to call it κ . Thus, what authors do or don't do can lead the mind and prepare it for what they want the student to accept in the future. This is an example of the primary purpose of textbooks: not to teach the results of a discipline but to supply the prevailing paradigm for the discipline. The second reason for this is to demonstrate the arbitrariness of some interpretations. In the above we could have just as easily associated κ with the number of units of charge that go around during T ; with the relative area (allowed orbital radii), velocity (phase and group velocities), or period of the charge to particle dynamics (comparing rotations in physical and "spin" space as is done here) as κa , κv , or T/κ respectively; or, as is usually done at this level, with an experimentally determined correction factor. κ may also be associated with \mathbf{L} itself. g is typically given a more sound theoretical footing using Dirac's relativistic formulation of quantum mechanics.

its orbit, described by $\hat{\mathbf{L}}$, there is only *half* a rotation of the charge that determines $\hat{\boldsymbol{\mu}}$ and corresponds to $\hat{\mathbf{S}}$. As it turns out, the fact that one full rotation in physical space corresponds to just half a rotation in spin space is exactly the result of other theoretical spin descriptions.

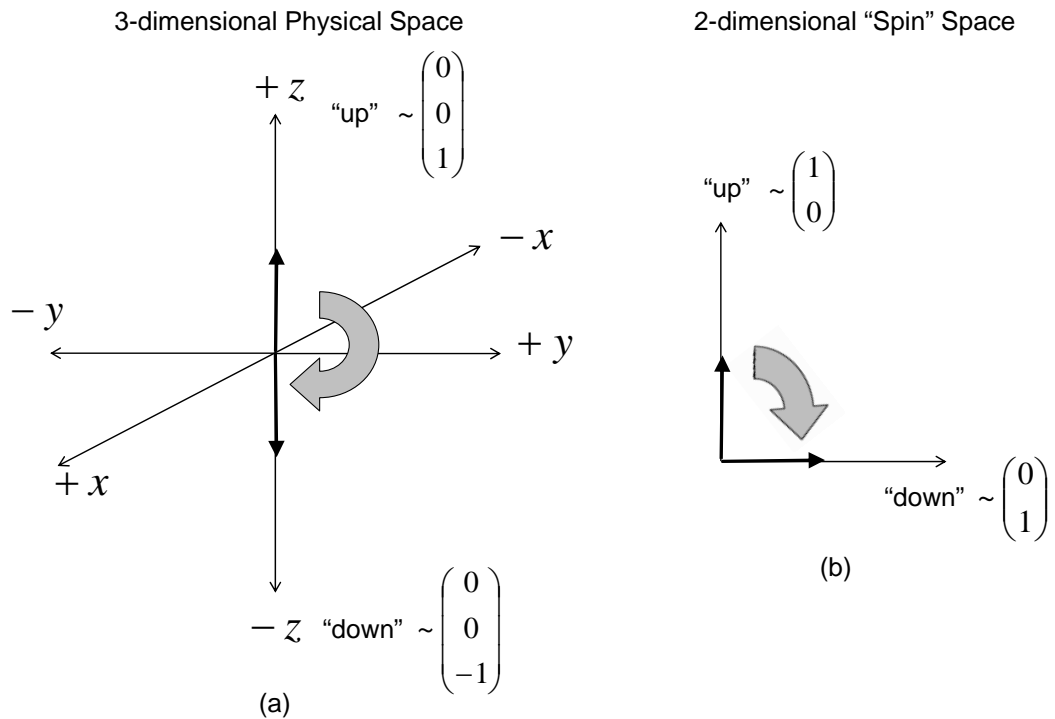


Figure 3.4: (a) Rotations of 180° in the familiar 3-dimensional, physical space correspond to rotations of only 90° in "spin" space. That is, the "up" and "down" directions in z are 180° apart in physical space but because they can completely describe all the possible outcomes of a spin- z measurement they completely span the space describing spin properties. Thus they can be thought of as an orthogonal basis of this space.

Also, in contrast with classical measures of angular momentum, $\hat{\mathbf{S}}$ can take on only $2s + 1$ discrete values where s is the quantum number defining the spin characteristics of a system. For example, a measurement of the spin of the electron, for which $s = 1/2$, in a given direction can only result as spin "up" or spin "down" in

that direction.⁶ Due to its two-valuedness, single electron spin systems are two-level systems. These are extremely useful for modelling more complicated systems of more states because they are the simplest systems that incorporate both the properties of a single state with the phenomenon of transitions *between* states. Consequently, whenever we refer to our system it is assumed to be a spin-1/2 system.

We can use these facts to define the operators corresponding to spin-1/2 systems. According to eq. (3.20) $s = 1/2$ means $N = 2$ so $\hat{\mathbf{S}}$ can be expressed with 2×2 matrices. Taking experimental results as constraints on the theory we often define the spin operator as

$$\hat{\mathbf{S}} = \frac{\hbar}{2} \hat{\boldsymbol{\sigma}} \quad (3.22)$$

where

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3.23)$$

are the Pauli matrices in the standard representation that arbitrarily diagonalizes σ_z .

We also represent \hat{H} as a 2×2 matrix. This necessitates representing any spin state $|\psi\rangle$ as a two component vector, or spinor,

$$|\psi\rangle \rightarrow \begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow} \end{pmatrix} = \psi_{\uparrow}\chi_{\uparrow} + \psi_{\downarrow}\chi_{\downarrow} \quad (3.24)$$

instead of as a scalar state as before. Also,

$$\chi_{\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_{\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.25)$$

are the eigenstates that diagonalize $\hat{\sigma}_z$. Thus the subscripts $\uparrow\downarrow$ respectively denote the spin as either “up” or “down” in the z-direction. Using eq. (3.20) we can write a general spin state with both time and space parts

$$\Psi = \psi_{\uparrow}\chi_{\uparrow}e^{-\frac{i}{\hbar}E_{\uparrow}t} + \psi_{\downarrow}\chi_{\downarrow}e^{-\frac{i}{\hbar}E_{\downarrow}t} \quad (3.26)$$

where $E_{\uparrow\downarrow}$ are the energies, i.e. the eigenvalues of \hat{H} , corresponding to the “up” and “down” states of $\chi_{\uparrow\downarrow}$.

⁶Perhaps this is easier to understand as spin “right” and spin “left” in the x or y directions but regardless of which direction we usually distinguish the two values as “up” and “down”.

3.2.2 Expectation Values

Important in any theoretical treatment of quantum measurements is the concept of expectation values. For a given state $|\psi\rangle$ every operator \hat{A} has an expectation value

$$\langle \hat{A} \rangle \equiv \langle \psi | \hat{A} | \psi \rangle. \quad (3.27)$$

Using the specific case that $\hat{A} = \hat{x}$ Griffiths [1] explains

[This] emphatically does *not* mean that if you measure the position of one particle over and over again, $[\langle \hat{x} \rangle]$ is the average of the results...Rather, $\langle \hat{x} \rangle$ is the average of measurements performed on particles *all in the state* ψ , which means you must find some way of returning the particle to its original state after each measurement, or else you prepare a whole ensemble of particles, each in the same state ψ , and measure the positions of all of them: $\langle \hat{x} \rangle$ is the average of *these* results (p. 14).

Discussing this further Griffiths [1] continues

this time in terms of velocity $d\langle \hat{x} \rangle / dt$, “Note that we’re talking about the ‘velocity’ of the *expectation* value of x , which is not the same thing as the velocity of the *particle* (p. 15).

With these statements we see that expectation values are averages of repeated measurements on identical systems not of repeated measurements on a single system.

3.2.3 Measurement

As one can see, measurement plays a significant role in both our practice and interpretation of quantum mechanics. There is however much ambiguity and debate as to what exactly the process of measurement entails. In attempts by Bohr, Von Neumann, Wigner, and others to clarify the ontological and/or epistemological nature of these issues some unfamiliar concepts such as the unpredictable “collapse” of the

wave function or the placement of a “cut” between the quantum and classically described worlds have been introduced.⁷ In what might be called the orthodox opinion “Observations not only *disturb* what is to be measured, they *produce* it...[When measuring position] we *compel* [the particle] to assume a definite position.” (Jordan in [1]) However, because of the proliferation of the ambiguous and unfamiliar ideas of the “cut” and “collapse” there is only a quasi-standard conception of what measurement is.

Because the SGE is considered the simplest and most clear demonstration of the measurement of quantum phenomena as opposed to classical expectations it is usually taken as the canonical, or defining, example of the process of quantum measurement.

3.2.4 The Uncertainty Principle

Another characteristic of quantum mechanics that must be brought up, and which we will bring up, in the discussion of measurement and the SGE is the uncertainty principle as formulated by Werner Heisenberg. It simply states that for a set of non-commuting operators, say \hat{x} and \hat{p}_x , whose commutator is

$$[\hat{x}, \hat{p}_x] = \hat{x}\hat{p}_x - \hat{p}_x\hat{x} = i\hbar \quad (3.28)$$

the relation can be given

$$\Delta\hat{x}\Delta\hat{p}_x \geq \frac{\hbar}{2}. \quad (3.29)$$

where the square uncertainty of a particular measurement \hat{A} is

$$\Delta\hat{A}^2 = \langle\hat{A}^2\rangle + \langle\hat{A}\rangle^2. \quad (3.30)$$

This is usually interpreted to mean that there is an irreducible uncertainty associated with the simultaneous measurements of \hat{x} and \hat{p}_x . In other words, the more confident we are of the result of a measurement of the position x of a particle, i.e. smaller $\Delta\hat{x}$, the less certain we are of a simultaneous measurement of its momentum

⁷For a standard discussion of measurement see [10]. For an expression of some concerns involved in these issues see [11] or [12].

in that same direction \hat{p}_x , i.e. $\Delta\hat{p}_x$ increases. To what extent this is an ontological or epistemological principle is unclear.⁸

Of particular interest to our discussion here is the realization that the operators, denoting measurements of the three orthogonal spin directions \hat{S}_x , \hat{S}_y , and \hat{S}_z , do not mutually commute. Their simultaneous existence as well defined quantities is therefore constrained by

$$[\hat{S}_i, \hat{S}_j] = i\epsilon_{ijk}\hat{S}_k. \quad (3.31)$$

More specifically, in any conceivable observation of the SGE no two components of the spin will ever be specified with more certainty than eq. (3.31) allows.

3.2.5 Stern and Gerlach's Experiment

Now that we have built up concepts relevant to the quantum representation of the SGE we can outline what might be its typical quantum derivation [9]. Such a derivation usually proceeds with the intent of making as few changes as possible to the classical account.

There are some stark differences however. For example, there is no widely accepted or technically defined concept of force in quantum mechanics, such as a force operator, so our approach to the SGE must be given in terms of quantum concepts and equations as given above. It is required to construct a Hamiltonian operator from the interaction energy in eq. (3.4). It is

$$\hat{H}_{interaction} = -\hat{\boldsymbol{\mu}} \cdot \hat{\mathbf{B}} \quad (3.32)$$

where all objects are now operators. Using eqs. (5.13) with $g = 2$ now, (5.14), and (5.15) with the field of eq. (3.7) we can write this interaction as a matrix operator.

$$\hat{H}_{interaction} = -\frac{e\hbar}{m} \frac{\hat{\sigma}_j \hat{B}_j}{2} = -\frac{e\hbar}{m} \frac{\hbar}{2} (B_0 + bz) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.33)$$

Deriving the results of the SGE is easiest if we express the full Hamiltonian in the frame of the beam. This sets the kinetic energy terms to zero. For simplicity we

⁸Again, inasmuch as [11] and [12] critique the measurement process they also raise several interesting issues involving the uncertainty principle.

also treat the field as an ideally impulsive field of duration T . In this case $\hat{H}_{interaction}$ becomes the full Hamiltonian for $0 \leq t \leq T$. The Hamiltonian for all times becomes

$$\hat{H}(t) = \begin{cases} 0 & \text{for } t < 0 \\ -\frac{e\hbar}{m} \frac{B_0 + bz}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{for } 0 \leq t \leq T. \\ 0 & \text{for } t > T \end{cases} \quad (3.34)$$

If we restrict ourselves to times such that $0 \leq t \leq T$ then \hat{H} is a constant in time and we may solve for its eigenvalues or energies. This can be done to get

$$E_{\uparrow\downarrow} = \mp \frac{e\hbar}{2m} (B_0 + bz). \quad (3.35)$$

There are two solutions labelled \uparrow and \downarrow because the system was describable by a 2×2 matrix. These are referred to as the spin ‘‘up’’ and the spin ‘‘down’’ states in the basis which diagonalizes $\hat{\sigma}_z$. So they mean ‘‘up’’ and ‘‘down’’ in z .

Using our previous results for describing a general state in terms of these eigenstates (eq. (3.26)) we may evaluate this state at $t = T$ and, after some rearranging, get

$$\Psi = \psi_{\uparrow} \chi_{\uparrow} e^{i\frac{e}{2m} B_0 T} e^{\frac{i}{\hbar} (\frac{ebT\hbar}{2m}) z} + \psi_{\downarrow} \chi_{\downarrow} e^{-i\frac{e}{2m} B_0 T} e^{\frac{i}{\hbar} (-\frac{ebT\hbar}{2m}) z}. \quad (3.36)$$

for the general state of the beam after emerging from the field at time T . This is valid then for all $t \geq T$.

If we compare this to the familiar form for an infinite plane wave travelling in the $\mathbf{k}=\mathbf{p}/\hbar$ direction with momentum \mathbf{p}

$$e^{\frac{i}{\hbar} (\mathbf{p}\cdot\mathbf{x} - Et)} \quad (3.37)$$

we see that there are two distinct momenta both in the z -direction. Namely,

$$p_z = \pm \frac{eb\hbar T}{2m}. \quad (3.38)$$

The \pm signs refer to the spin ‘‘up’’ and ‘‘down’’ states respectively so the two orientations of spin are sorting themselves along the z -direction in a discrete manner. Notice

that it is proportional to the field gradient b . If a beam of particles with randomly oriented spins is subject to a magnetic field possessing both homogeneous and inhomogeneous parts the beam emerges with all particles either with spin “up”, as defined by the uniform field, and with a momentum proportional to the gradient, or with spin “down” and travelling with the same momentum but in the opposite direction. This is because the spin behavior $\chi_{\uparrow\downarrow}$ is entangled with the spatial behavior $\pm p_z$. Thus, on a screen located behind the apparatus the general states eq. (3.36) collapse into two distinct traces indicating respectively the spin “up” and spin “down” states along the preferred field direction. This is in stark contrast to the classically expected result in which a single continuous trace appears.

Apart from clearly showing the quantized nature of spin and the divergence of quantum mechanical concepts from classical ones the SGE is taken as the canonical example of the quantum process of measurement. This gives it an important place in the history and development of quantum mechanics.

Chapter 4

A Historical Account of the Stern-Gerlach Effect

In the previous chapter we built up a description of the SGE from fundamental concepts. This is not the only way to represent the story however. Just as we can represent a function in terms of different coordinates systems (see fig. 2.3) we can also portray the development of the SGE in different ways. Each will emphasize a different aspect of the story. The historical account of the SGE, which is given here, though perhaps not found as universally in textbooks, is extremely valuable on its own.

The historical account given in this chapter does not present all the detail that could be given due to practical constraints.¹ We give here only enough detail to capture, in the end, the general nature of historical representations and, in particular, the divergence of this account of the SGE from the thematic one given previously.

4.1 Atomic Models

Prior to 1900 classical mechanics was the prescribed methodology for progress in physics. It provided the most universally accepted and powerful paradigm in physics. It had begun in embryonic form with the revolution of Copernicus but was carried on by others such as Kepler, Galileo, Descartes, and set in full motion by Newton. To a large degree, since that time, physics has consisted of the working out of the implications of Newton's laws of motion. In the Kuhnian terminology of

¹The careful reader will realize that this statement implies that the historical account here, and in fact historical accounts in general, are really specific types of thematic accounts. This is similar to our previous note that mathematical representations are a subclass of the more general conceptual ones. In other words, we select in chronological order only those events which we determine as appropriate to the historical themes we want to convey.

chapter 1 and [2], in Newton had culminated a shift of paradigm and most subsequent physics consisted of casting observed data in the paradigm-provided mold and not in creating the mold itself.

One phenomenon of interest during this period of “normal” science was the description of the atom. In the late 1800s a debate existed between those that thought nature was fundamentally continuous and those that considered it fundamentally discretized. These former proponents were the atomists. But Einstein’s work on Brownian motion in 1905 was considered the first experimental demonstration of atomic behavior. Up to this point talk of atoms had been only theoretical and based on macroscopic secondary effects. Under the classical regime, atomic behavior, as every other phenomenon, was thought to strictly follow Newtonian laws. With the demonstration of Brownian motion showing the discreteness of atomic particles, and other developments including but not confined to Einstein’s other 1905 paper concerning the photoelectric effect and an earlier purely theoretical description of black body radiation by Max Planck in 1900 in which a completely *ad hoc* factor h was introduced [14] the paradigm of discreteness, or quantization, gained widespread acceptance.

Accordingly, under the extant models had anyone proposed the SGE *at this time*, the theoretical description would have been similar to that which was given in section 3.1. It is important to realize however that the SGE was not even conceived of until much later, after other developments had occurred.

4.1.1 1913: The Bohr Model

From extensive spectroscopic measurements it was concluded that atoms of a particular type always seemed to emit the same definite and distinct amounts of energy. For example, when observing the light emitted from a tube of gas that had been excited with an electrical voltage the same spectrum of colors always appeared. Even more interesting was the fact that this spectrum was discrete.

In 1913 Niels Bohr developed a model of the atom that mathematically and conceptually captured this discrete behavior. It resembled, though not exactly, the familiar picture of the solar system with particles moving around the nucleus in

various coplanar, circular orbits of *fixed but discrete* radii. These radii determined the energies of the atom like gravitational potential energy in a solar model.

The discrete nature of the orbital radii also discretized the magnitude of the angular momentum \mathbf{L} because of its dependence on the radius. Accordingly, in order to agree with experimental findings Bohr asserted that

$$L = n\hbar, \text{ with } n = 1, 2, 3, \dots \quad (4.1)$$

where h is Planck's recently introduced constant [15].

Because of its now outdated quantization rules and picture Bohr's scheme is known today as the "old quantum theory."

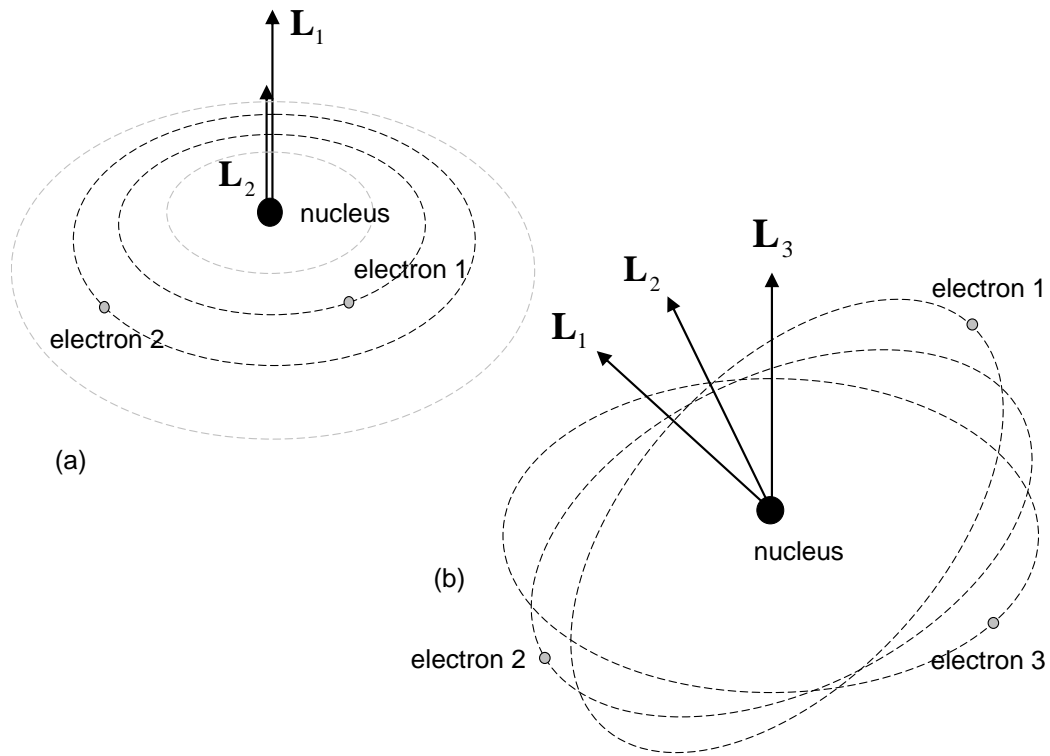


Figure 4.1: (a) In the Bohr atom the electron has a definite trajectory that can only occur at certain coplanar distances from the nucleus. Thus the magnitude L can likewise be discrete or quantized. (b) In Sommerfeld's extension of Bohr's ideas the electron paths could occur in different planes though they were still constrained to quantized radii. Because this allowed for the additional quantization \mathbf{L} in its *spatial* orientation this is referred to as space quantization.

4.1.2 1916: The Sommerfeld Model

As with any model however the Bohr model of the atom did not fully describe all the details of observed atomic phenomena. In 1916 Arnold Sommerfeld aided in extending the Bohr atomic model to other cases including relativistic effects and the quantization of all three components of \mathbf{L} . In doing so the Bohr's conceptual representation was altered in a few ways.

The circular coplanar orbits that were visualized in the Bohr atom were replaced with orbits that could be distorted to elliptical shapes and could take on various orientations. They did not have to be coplanar. As a result Bohr's makeshift result of a quantized \mathbf{L} in *magnitude* was extended to also include the possibility of quantizing the *direction* of \mathbf{L} as well.

4.2 1921-1922: Stern and Gerlach's Experiment

We stop here with the story of atomic models because this is the environment in which Otto Stern and Walther Gerlach found themselves in 1921. This was the model - or paradigm - with which they were working.

It was Stern and Gerlach's intent to verify the Bohr-Sommerfeld model of the atom by measuring the quantized states of \mathbf{L} . As we have seen, based on their "old" quantum intuition Stern and Gerlach assumed that the atom possessed angular momentum made manifest in the orbit of the electron about the nucleus. This implied the presence of a magnetic moment $\boldsymbol{\mu}$ which could be manipulated via a magnetic field \mathbf{B} . There were several considerations that would have been either explicitly confronted or unknowingly passed by. We list a few of these based on what seems appropriate to us for our purposes.²

4.2.1 Magnet Type

The first consideration may have been of the particular magnetic field configuration that would be used. As we saw in the classical picture of section 3.1 if

²The following may or may not have been exactly how these issues played out in the minds of Stern and Gerlach but only *represent* how those issues may have been resolved in the given historical context.

Stern and Gerlach wanted to observe the magnitude of μ *in a particular direction* there were two essential components to the field. They needed (1) a non-uniform component to \mathbf{B} so as to cause the force differential needed to sift the particles by an observable amount. This amount may have been determined by diffraction considerations, charge, or mass. And (2) Stern and Gerlach needed a preferred direction to \mathbf{B} in order to define the component of μ being measured. In section 3.1 this was done with the introduction of B_0 .

Spatial variations of the field had to be considered as well. Once generated with an appropriate momentum, towards the detector, the particles had to enter and exit the field. Considered in the frame of the particles this would introduce the same effects as a time-dependent field. Thus maintaining the desired axial uniformity as well as avoiding unwanted dynamic effects would have to be considered in order to make the results clear.

4.2.2 Particle Choice

One way of avoiding several issues with the fringe field effects was to chose a very specific type of particle. Following Maxwell's equations this changing \mathbf{B} -field would create an electric field that could exert Lorentz forces on particles carrying charge. These forces could easily blur the beam in unintended directions disguising the outcome and interpretation of the experiment.

Stern had worked with beams of silver atoms before so this was the natural choice. They are electrically neutral but still possess a magnetic moment. That is, in their neutral state they carry as many protons as electrons but only have one valence electron so that, while all charge cancels out, there is an "extra" electron orbiting the nucleus overall thus creating a magnetic moment. These atoms constituted the beam.

4.2.3 Beam Width

The atoms of the beam were boiled off a filament and accelerated towards the magnets for measurement. This process imparts a random distribution of both

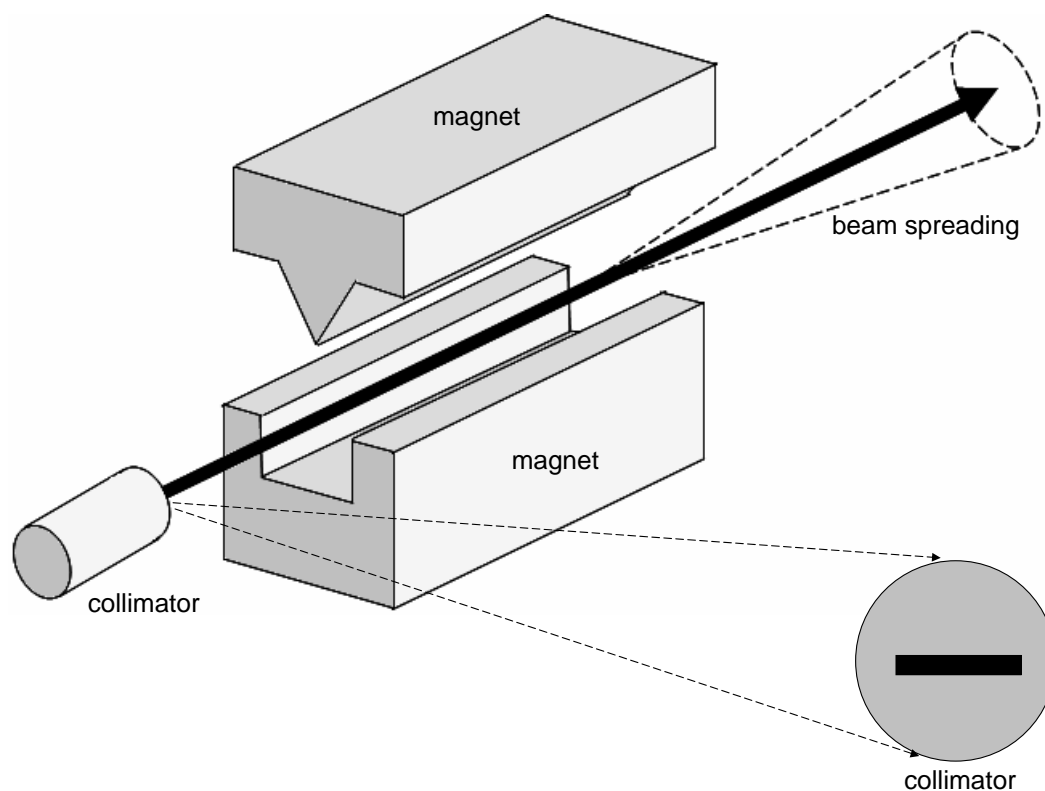


Figure 4.2: After leaving a collimator the Stern-Gerlach beam entered a magnet with a notch cut into one side and a point on the other. The resulting non-uniform field caused the beam to separate in different directions due to what is now thought of as spin. It was detected on a pane of glass.

momenta and magnetic moment to the particles of the beam. The magnetic moments were considered random only in direction. This is exactly what the experimenters wanted in order to measure the nature of the distribution of directions. However, the atoms had to be selected according to their momenta in order to direct them at the measuring apparatus. This was done by collimation.

Stern and Gerlach chose to collimate the beam using a long narrow slot oriented perpendicularly to the axis along which the measurement was to be taken and the beam axis. This gave them a very narrow beam in the direction of interest but yielded a very diffuse beam in the perpendicular direction fig. (4.3). As they were only concerned with one direction - the direction selected by the magnetic field - this was sufficient.

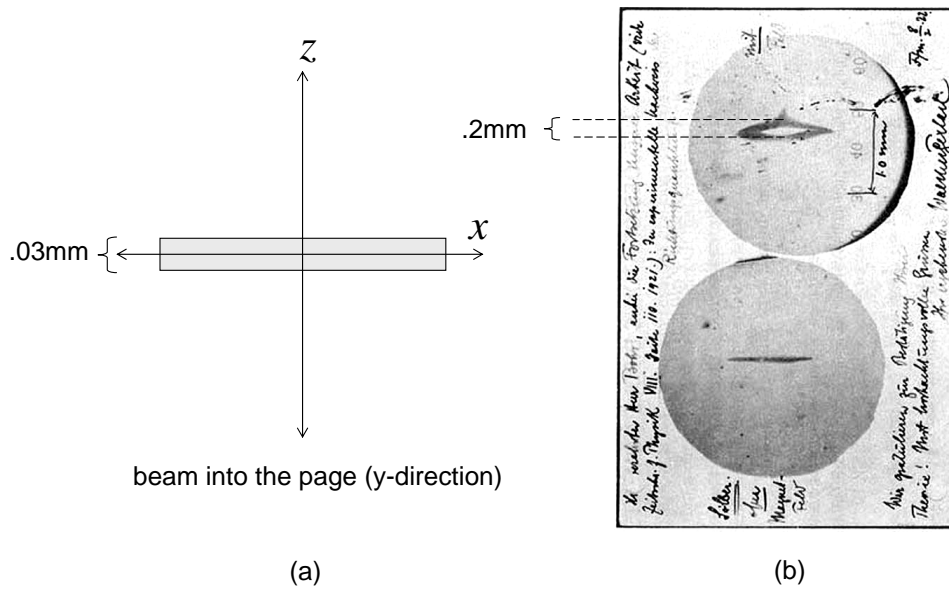


Figure 4.3: (a) The cross section of the Stern-Gerlach beam in the yz -plane was long and narrow in the xz -plane. (b) The actual images taken from the post card Gerlach sent to Bohr announcing his results. There is clearly a discrete separation in one direction. Taken from [17].

4.2.4 Results

After sending the beam through the poles of the chosen magnet it proceeded to a glass plate for detection some distance away where it left a deposition. After several attempts at fine tuning the apparatus and the intervention of a cheap cigar [16] a trace was recovered that gave Stern and Gerlach *exactly what they had predicted from theory* - a definite separation into two distinct traces oriented along the direction of the uniform field fig. (4.3).

4.3 1925-1926: Quantum Mechanics

To this point the SGE had been spoken of in terms of mostly classical concepts and equations. *At most* it participated only in substantiating the *old* quantum theory of the Bohr-Sommerfeld atom. The fundamental equations of modern quantum mechanics had not even been developed yet. It wasn't until 1925 that Heisenberg

developed his matrix formulation for the fundamental characteristics of quantum mechanics with his equation

$$i\hbar \frac{\partial}{\partial t} \hat{A} = [\hat{A}, \hat{H}] + i\hbar \frac{\partial}{\partial t} \hat{A} \quad (4.2)$$

in which the arbitrary operator \hat{A} and the Hamiltonian \hat{H} are the relevant objects. It was not until the next year that Schrödinger proposed his more widely recognizable wave mechanics formulation using the then still mysterious wave function $\psi(\mathbf{x})$.³ His famous equation

$$i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2m} \nabla^2 \Psi + \hat{V} \Psi \quad (4.3)$$

is the time-dependent Schrödinger equation. It is typically found in textbook discussions of quantum mechanics and also used in the previous chapter.

4.4 1925: Spin

The development of these two formalistic systems, the Heisenberg and Schrödinger pictures, signaled our entrance into a new era of physical science. We were undergoing a shift of paradigm. With each new unexpected result - and there were many - we were having to define and redefine the conceptual basis upon which we could build our theories and against which we would push off when we once again felt equipped to return to “normal” science.

Once such result that required the introduction of a new concept for its proper placing in the framework arose from atomic physics. The spectral lines of a given element could be observed to bifurcate into two closely spaced identical lines when the element was placed in an extremely strong external magnetic field. This was the Anomalous Zeeman effect. The word that expressed the concept that was needed to connect this behavior to others in the quantum framework was coined by two graduate students, Samuel Goudsmit and George Uhlenbeck, in 1925. It was spin.

It appears that the quantum mechanical concept of spin wasn't associated with the SGE of 1922 until 1927.[18]

³Max Born gave ψ its present interpretation as a probability amplitude.

4.5 1935: Measurement

During the years under discussion here not only was the scientific community working out the technical implications of the new quantum regime but they were simultaneously attempting to define its conceptual foundations. As we have pointed out, one of the difficult phenomena to translate was that of measurement.

The exact timing of the designation of the SGE as the clearest experimental demonstration of the measurement problem is not well defined. From a historical perspective the phenomenon of measurement didn't force itself into the forefront of scientific philosophical discussion until 1935 when Einstein, Podolsky, and Rosen published a paper challenging the then prevalent byproduct of this quantum conceptual revolution.⁴ Their proposal came to be known as the EPR paradox because of its clear and fundamental repudiation of familiar ideas. It placed the quantum concepts of entanglement and measurement in the forefront of our framework. Perhaps because these two concepts are extremely clear and easy to represent in a discussion of the SGE, as we have shown in the previous chapter, simultaneously with a clearly distinct classical analogue, the SGE has consequently been given important status as well. For this reason it is important to address the problems in our descriptions of the SGE.

⁴Although the EPR paper was published in 1935 debates and controversy regarding other aspects of the proper interpretation of quantum mechanics had carried on for as many as 10 years. The discussions of the Solvay conference of 1927 are particularly interesting.

Chapter 5

Problems With Accounts of the Stern-Gerlach Effect

We have given both a thematic, or textbook, account of the SGE and a historical account of the same. The former is built up by the logical ordering necessary to systematically construct appropriate concepts and the latter by the *chronological* ordering of human experience. As representations both presentations emphasize different aspects of the storyline. They also obscure other characteristics either inadvertently or because they are explicitly deemed less valuable. In the following we discuss the assumptions and inconsistencies that are hidden in the two previous accounts but that are rashly dismissed either because of tradition, practicality, or misunderstanding. Before considering the limitations of these two accounts however we will consider some limitations that appear to apply more generally to quantum mechanics, and even science, as a whole.

5.1 Problems from Quantum Mechanics

In section 2.3.3 we introduced wave-particle duality as an example of orthogonal concepts. We called such concepts “orthogonal” because, like orthogonal vectors or functions, the concepts implied by one cannot in any way be represented in terms of the concepts implied by the other. In the language of logic the two are mutually exclusive. In this sense orthogonal concepts are also describable in terms of Bohr’s principle of complementarity in which two disparate concepts are required in order to fully describe a single phenomenon (see discussion in section 2.2).

5.1.1 Rightness and Clarity

Among the many of concepts that serve as effective conceptual basis sets for describing any scientific concept are the concepts of rightness and clarity. They are an effective basis because they are “orthogonal” or complementary. As applied to

Some Orthogonal Concepts	
Rightness	Clarity
Waves	Particles
Position	Momentum
Justice	Mercy
Creativity	Conformity
Humility	Confidence
Patience	Ambition
Form	Function
Constancy	Change
\hat{x}	\hat{y}

Table 5.1: A listing of some orthogonal concepts to which Bohr applied his principle of complementarity. Just as with coordinate axes it is precisely because of their orthogonality that they are useful in describing other, more complicated, ideas. [19]

representations we can see that the more effort that is put into making a presentation simple, clear, and accessible the more idealization, approximation, and artificiality are in it. Whereas if a strenuous effort is made to represent all the facts in their proper context so much complication and technicality are introduced that a useful understanding of them becomes near impossible. In short, the more clear a representation is made to be the less right it is whereas the more right it is the less clear it can be. Thus there is a fundamental tension between the rightness and clarity of a statement.

5.1.2 Representations as a Map

This can be demonstrated by considering a map. Maps are representations of regions of space. The more detailed the map the bigger and bulkier must be its pages with expanded scaling and legends whereas the more compact and simple the map the less detail can be described by it. To maximize its correspondence to reality at the expense of user-friendliness would make the map no different than the terrain it describes while increasing its immediate and efficient use would surely omit some detail that could become important.

5.1.3 Communication

The simultaneous maximization of both the clarity and rightness of a statement is the aim of effective representation and communication. However, in quantum mechanics this discrepancy between what *is* and what we *understand* is manifest in even stronger terms than in other fields. It was a common opinion among the architects of quantum theory that although quantum phenomena were incompatible with classical concepts, due to the classical nature of the equipment, i.e. its compatibility with humans on the macro-scale, only these concepts could be used to describe them. For example, Bohr [20] has said that

In this context, we must recognize above all that, even when the phenomena transcend the scope of classical physical theories, the account of the experimental arrangement and the recording of observations must be given in plain language. (p. 72)

and Heisenberg [21] has written that

Any experiment in physics, whether it refers to the phenomena of daily life or atomic events, is to be described in the terms of classical physics.
(p. ??)

In other words, the common opinion is that although nature fundamentally behaves quantum mechanically humans can only understand it in terms of classical concepts

because we can only interact with this level.¹ Unfortunately for us, classical concepts are also inadequate.

Thus, it is a frustrating axiom of modern paradigm that *clear* communication is exactly opposed to *correct* communication. As one aspect is refined or improved the other is helplessly compromised. This may be suggestively expressed in the schematic form

$$\Delta\text{Clear}\Delta\text{Right} \geq \text{constant}, \quad (5.1)$$

reminiscent of eq. (3.31).

When students find this out for themselves it is not unlike entering hell in the *Divine Comedy*.²

5.2 Problems with Historical Accounts

In terms of understanding the SGE some historical details might be missed - either deliberately or ignorantly - in the historical accounts but of more concern for us here is their lack of clarity. Consequently we will not spend time here to discuss the problems of historical accounts as regards to facts but only emphasize their pedagogical limitations.

There are some of the concerns common to these sorts of accounts.

More historical detail, whether of science's present or of its past, or more responsibility to the historical details that are presented, could only give artificial status to human idiosyncrasy, error, and confusion. Why dignify what science's best and most persistent efforts have made possible to discard? [2]

Indeed, including all the facts associated with each "dead end" pursued by each researcher is impractical and surely confusing to students at an introductory level. At this point students need a clear, linear progression that builds the concepts in a logical way so as to make them as graspable as possible. Unfortunately this is not

¹For a similar discussion see also [22] p. 127-130.

²In Dante's classic *The Divine Comedy* the inscription above the doorway leading to hell read "Abandon hope, all ye who enter here."

how history typically unfolds. The development of science is often more messy than it is linear - running into several problems followed by some backtracking before making a small but successful advance. Sometimes paths are vigorously pursued only to show they are fruitless.³ Thus, in the attempt to make the story of the SGE correspond to the reality of the events, which for some purposes is a worthy goal, for instructional purposes clarity is lost.

5.3 Problems with Thematic Accounts

On the other hand, the strength of thematic accounts of scientific events is precisely their clarity. However, according to the dual relationship hinted at in eq. (5.1) this implies only a crude correspondence to fact or reality. As the bulk of this work will assess the accuracy of thematic accounts with a special emphasis on their technical aspects (see the next chapter, in particular) we will spend the remainder of this chapter developing these ideas.

5.3.1 An Artificial History

First, in a general sense opposite to that of section 5.2 the thematic accounts of textbook or classroom discussions are very linear and cumulative in their presentation and use of concepts. That is, before a particular concept is needed in the description of an event that concept is either motivated or derived from previous concepts. This of course removes concepts out of their historical context and therefore assigns them an artificial and unscientifically acquired meaning. This also necessitates the eventual acceptance of certain axioms which ultimately have no justification but are deemed useful.⁴ However, convenience is too often confused with correctness. These considerations lead us to the conclusion that there is little that corresponds to reality in these accounts. Their final justification is only that they are clear and that they

³The word “fruitless” here is a misleading word. While such a path may be “fruitless” as regarding the *desired* result, inasmuch as an understanding of what definitely will not work is gained this is a vital step *forward* in producing progress.

⁴For this reason [2] tells us that it is the purpose of textbooks to communicate paradigms and not to communicate facts.

posses an accurate predictive power. This says little, however, about the accuracy of the conceptual structure that had to be built up.

From contrasting the thematic account in Chapter 3 with the historical one of the Chapter 4 we can surmise that there are several misconceptions that students could have after having been taught the SGE.

(1) The original SGE was observed several years *prior* to the development of the quantum theory and spin. Hence, although it is often used as a confirmation or demonstration of modern quantum concepts such as spin and measurement *it was in no way motivated by or intended for this purpose.*

(2) The observation of the SGE actually *confirmed* the historically prevalent theory of the time - the Bohr-Sommerfeld model of the atom - *which is now considered false.*⁵ It did not surprise the experimenters for this is exactly what Stern had predicted previously [23]. Those doors had either already been open, such as with the Bohr model's quantization, or would not be open for years, such as the incorporation of spin into the quantum framework. Therefore, the SGE did not signal the opening of a new door in physics or philosophy but the relatively unexciting continuance of the present paradigm.

(3) It is interesting to note that the SGE clearly demonstrates how the right results⁶ - even experimental results - can be used to justify an erroneous account of the phenomena and *then also be given canonical and defining status in a completely new conceptual system.* This demonstrates the subjectivity of representations in science well.⁷

[2] describes some of the general deficiencies of thematic approaches as well. It says,

The result [of the thematic approach of textbooks] is a persistent tendency to make the history of science look linear or cumulative... The

⁵On a post card communicating the results of their experiment to Bohr Gerlach writes "We congratulate [you] on the confirmation of your theory." See [17]

⁶This is clumsy language. Results can never be wrong especially in experimental physics; only our questions or our interpretations of the results can be difficult.

⁷This is similar to the researcher who can always seem to get the answer he's looking for precisely because he's looking so hard for it.

textbook tendency to make the development of science linear hides a process that lies at the heart of the most significant episodes of scientific development. (p. 137-140)

It hides the true nature of the most interesting research questions.

5.3.2 Misunderstanding the Practical Aspects of the Stern-Gerlach Effect

Related to misconceptions as to the historical ordering and significance of events are the practical experimental considerations which we included in sections 4.2.1-4.2.3. They concern (1) the species of particle in the beam, (2) selection of beam cross section, and (3) the type of magnetic field used.

The usual de-emphasis of these three points in thematic accounts of the SGE has some interesting consequences.

(1) (See section 4.2.1) Because all the talk of the SGE is couched in terms of spin-1/2 particles the idea, if not the word explicitly, of the electron is used. It is considered the canonical spin-1/2 particle. However, this practice hides a whole field of very interesting research. Students don't realize that there is a time-dependence to the magnetic field due to its approaching speed relative to the particle which exerts a transverse Lorentz force on it. Because this phenomenon is dependent on the presence of charge *the SGE has never been observed with electrons*. In fact, Bohr and Pauli, among others were of the opinion that the electron SGE could *never* be observed [24].

This points out an inconsistency. As a canonical example of spin-1/2 particles electrons should display the SGE as do other spin-1/2 particles like silver.

Despite this, there is much research going on to overcome these blurring effects with fruitful results. Because this whole field opens more questions than it answers, which compromises clarity, it is often neglected.

(2) (See section 4.2.2) The astute reader might find an inconsistency between what is said and what is shown. Typically when the idealized SGE is discussed in textbooks it is spoken of in terms of a point-like beam for reasons of clarity. When the original trace of the experiment is also shown as in fig. 4.3 this is inconsistent. If both of these are taken as accurate then it seems there was a continuous blurring in the

horizontal direction that accompanied the discrete separation in the field direction. If this is not pointed out it can be confusing whereas if it is it incorporates many unnecessary experimental details that make its description bulky and difficult.

The previous point is especially important when considering the SGE as a demonstration of quantum measurement because it is accepted that the *entire* experimental context defines the phenomenon. In other words, if this is not made clear then it is unclear as to what we are really measuring. Students may be left wondering if a point-like SGE ever been attempted and if not, why not? This is difficult to determine from thematic accounts because so little is ever said about the specifics of the collimation.

(3) (See section 4.2.3) As a discussion of the misconceptions regarding the specifics of the magnetic field introduces the more technical aspects of thematic accounts and has lead to numerous questions it is treated in its own section.

5.4 The Magnetic Field

As was argued in section 3.1 the magnetic field

$$\mathbf{B} = (B_0 + bz)\hat{z} \quad (5.2)$$

is the simplest field to meet the physical requirements of the question that Stern and Gerlach sought to answer. In fact, this is a typical representation of the field as given in many discussions of the SGE. It is rarely mentioned however, especially in explicit terms, that there is at least one other physical constraint on the field.

Maxwell's equations for all electromagnetic fields specify that \mathbf{B} must satisfy

$$\nabla \cdot \mathbf{B} = 0 \quad (5.3)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{E} \quad (5.4)$$

where \mathbf{E} is the electric field and c is the speed of light in vacuum. μ_0 is the magnetic permittivity of vacuum. Because of eq. (5.3) \mathbf{B} must be inhomogeneous in *at least* two directions

$$\nabla \cdot \mathbf{B} = \frac{\partial}{\partial x} B_x + \frac{\partial}{\partial y} B_y + \frac{\partial}{\partial z} B_z = 0. \quad (5.5)$$

So a better choice would be

$$\mathbf{B} = -bx\hat{x} + (B_0 + bz)\hat{z} \quad (5.6)$$

because

$$\frac{\partial}{\partial x}B_x = -\frac{\partial}{\partial z}B_z. \quad (5.7)$$

As a fortunate consequence $\nabla \times \mathbf{B} = 0$ so in accordance with eq. (5.4) there are no dynamically changing fields. This is the simplest magnetic field that satisfies the conditions of Stern and Gerlach's question - it has homogeneous and inhomogeneous parts - *as well as* Maxwell's physical constraints. For reasons of clarity the field is often not treated this way. This is however the difference between a description of the SGE that is consistent with the fundamental equations of electricity and magnetism, and one that is unphysical.

It should also be pointed out that there are other constraints to further define the field. One that may prove important later is the physical requirement that the field remain finite at all distances. As it is now, the field linearly blows up with increasing distances away from the origin. That is,

$$\lim_{r \rightarrow \infty} \mathbf{B} = \infty \quad (5.8)$$

where $r = \sqrt{x^2 + z^2}$. To get around this we can either append the field with an exponential factor that enforces the required asymptotic fall off or merely truncate our range of interest to a finite region. The second of these is more artificial than the first. These considerations will be taken up in more detail in section 6.1.1.

5.5 Precession Arguments

Notwithstanding the inaccuracies in the field of eq. (5.2), as we have shown, it can still yield correct results. This indicates that there must be some physical justification for neglecting it. When the necessary complication of the full field is used it is the phenomenon of precession that is said to justify the neglect of the transverse inhomogeneity. We will consider how this is done in both the classical and quantum regimes.

5.5.1 Classical Precession

We have shown in section 3.1 and it has been more rigorously demonstrated in [25] and [26] that the presence of the homogeneous field component \mathbf{B}_0 causes a precession of the classical vector $\boldsymbol{\mu}$. From the torque equation

$$\frac{d}{dt}\boldsymbol{\mu} = \boldsymbol{\omega} \times \boldsymbol{\mu} \quad (5.9)$$

given there $\boldsymbol{\mu}$ can be found and substituted into the force equations. [25] and [26] show how when the time average of the force is calculated only the force in the z-direction turns out to be non-zero. From this fact it is argued the inclusion of precession *in the right way* justifies the neglect of all x-oriented dynamics.

It is interesting to consider what the “right way” consists of. Even in this classical picture the approximations are crude. As [26] explicitly points out the required relation is not just that there is a homogeneous component to the field but that

$$|\mathbf{B}_{homogeneous}| \gg |\mathbf{B}_{inhomogeneous}| \quad (5.10)$$

or, in the case of eq. (5.6)

$$|B_0| \gg |br| \quad (5.11)$$

where $r = \sqrt{x^2 + z^2}$. What is more subtle is that [26] then applies this particular criterion with the approximation that

$$\mathbf{B} \approx B_0 \hat{z}. \quad (5.12)$$

In other words, this inequality eq. (5.11) is only useful if it justifies the complete neglect of the inhomogeneity. So, in the classical treatment of [25] and [26], the field necessary to induce adequate precession, i.e. a strong uniform field, is exactly the field that prohibits the separation of spins by a net force, i.e. an inhomogeneous part. This is crude at best and inconsistent at worst.

There is one further condition on the experimental set up that is required for practitioners of classical physics to invoke the precession argument. Conditions must be chosen such that the time of interaction, or the time it takes the particle to pass through the region of space in which the field is significant, must be much greater

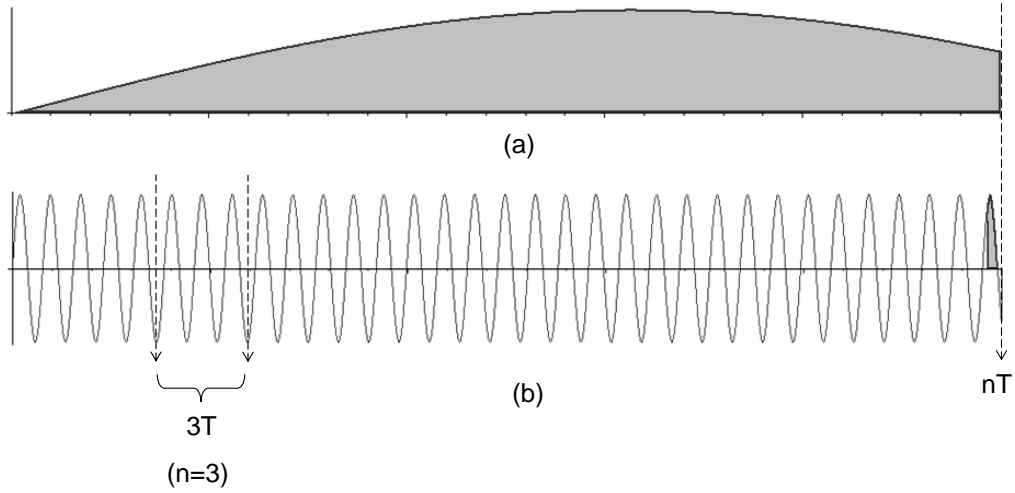


Figure 5.1: (a) If the time of interaction nT is roughly equivalent to or less than the period T of the precession frequency, that is $n \leq 1$, then the net area between the precessing curve and the axis is large. (b) If $n \gg 1$ such that several precession cycles occur during the time of interaction nT then the portion of the shaded region that does not cancel out is much smaller. In other words, for the same nT the total shaded region for a slowly precessing spin is much larger than that for a rapidly precessing spin. Inasmuch as the shaded region corresponds to the accumulated affect of the transverse force in time, i.e. its time-average, the transverse deflection of the beam only washes out in (b). For (a) we would still expect a transverse deflection.

than the period of precession. This allows the x-directed forces to adequately “wash out.” Without this the average x-force would tend to favor one side or the other.

5.5.2 Quantum Precession

In thematic accounts in which the quantum version of the precession argument is used to justify the use of a non-Maxwellian field configuration it is the expectation value of the spin that precesses and not the spin itself. This is an extremely subtle point.

Note that in the present context we use “spin” $\hat{\mathbf{S}}$ instead of “magnetic moment” $\hat{\boldsymbol{\mu}}$ according to their relation

$$\hat{\boldsymbol{\mu}} = \frac{ge}{2m} \hat{\mathbf{S}}. \quad (5.13)$$

Recall also that

$$\hat{\mathbf{S}} = \frac{\hbar}{2} \hat{\boldsymbol{\sigma}} \quad (5.14)$$

where

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (5.15)$$

as before.

Precessing Expectation Values

In order to show this we place a particle with spin 1/2 in a uniform \mathbf{B} -field in the z-direction

$$\mathbf{B} = B_0 \hat{z}. \quad (5.16)$$

Using this in the Hamiltonian

$$\hat{H} = -\hat{\boldsymbol{\mu}} \cdot \hat{\mathbf{B}} \quad (5.17)$$

we can find the general states and energies as in section 3.2.1

$$\Psi = \psi_{\uparrow} \chi_{\uparrow} e^{-iE_{\uparrow}t/\hbar} + \psi_{\downarrow} \chi_{\downarrow} e^{-iE_{\downarrow}t/\hbar}. \quad (5.18)$$

With these states, the expectation value of $\hat{\mathbf{S}}$, or more specifically $\langle \hat{S}_x \rangle$, $\langle \hat{S}_y \rangle$, and $\langle \hat{S}_z \rangle$, can be evaluated. First we write

$$\langle \Psi | \hat{S}_j | \Psi \rangle \quad (5.19)$$

where

$$\Psi = \psi_{\uparrow} \chi_{\uparrow} e^{-iE_{\uparrow}t/\hbar} + \psi_{\downarrow} \chi_{\downarrow} e^{iE_{\downarrow}t/\hbar}. \quad (5.20)$$

So, if $j = x$

$$\langle \hat{S}_x \rangle = \langle \Psi | \hat{S}_x | \Psi \rangle = \frac{\hbar}{2} \left(\psi_{\uparrow}^* \psi_{\downarrow} e^{\frac{i}{\hbar}(E_{\downarrow} - E_{\uparrow})t} + \psi_{\downarrow}^* \psi_{\uparrow} e^{\frac{i}{\hbar}(E_{\uparrow} - E_{\downarrow})t} \right). \quad (5.21)$$

Note that all the time-dependence is carried by factors that depend on the difference of the energy of the two states, more specifically $(E_{\downarrow} - E_{\uparrow})$ and $(E_{\uparrow} - E_{\downarrow})$.

Applying this procedure to \hat{S}_y and \hat{S}_z as well we get

$$\langle \hat{S}_x \rangle = \frac{\hbar}{2} \sin \theta_0 \cos \left(\frac{eb}{m} t \right) \quad (5.22)$$

$$\langle \hat{S}_y \rangle = \frac{\hbar}{2} \sin \theta_0 \sin \left(\frac{eb}{m} t \right) \quad (5.23)$$

$$\langle \hat{S}_z \rangle = \frac{\hbar}{2} \cos \theta_0 \quad (5.24)$$

where θ_0 is the angle the spin vector makes with the z-axis. Thus, as time progresses $\langle \hat{S}_z \rangle$ is constant and $\langle \hat{S} \rangle = \langle \hat{S}_x \rangle \hat{x} + \langle \hat{S}_y \rangle \hat{y}$ rotates, or precesses, in the xy-plane.

To clearly see the effect of this precession on quantum expectation values we take the time average. In general if we wanted to find the time average of some oscillating function $f(t)$ where the frequency of oscillation is $\omega = 2\pi/T$ over n periods T . We would evaluate

$$f(t)_{avg} = \frac{1}{nT} \int_0^{nT} f(t) dt \quad (5.25)$$

Applying this averaging formula to the expectation values above we see that

$$\langle S_x \rangle_{avg} = 0 \quad (5.26)$$

$$\langle S_y \rangle_{avg} = 0 \quad (5.27)$$

$$\langle S_z \rangle_{avg} = \frac{\hbar}{2}. \quad (5.28)$$

Thus, over significant time intervals the expectation values $\langle \hat{S}_x \rangle$ and $\langle \hat{S}_y \rangle$ “wash out,” or go to zero, whereas $\langle \hat{S}_z \rangle$ is constant.

Problems with Precession

In the standard description of the SGE this “washing out” effect of the expectation values is referred to in order to justify the neglect of the transverse inhomogeneity of eq. (5.6) and the use of a divergenceless field eq. (5.2). Some merely cite the classical case of the phenomenon as justification [27] while some refer to the fully quantum demonstration given above [1], [9]. Even in the latter case however, when the demonstration is explicit, this is inconsistent for at least three reasons.

(1) The above derivation depended critically on the value of the energies or, more specifically on the difference of the energies. Therefore, in order to rigorously justify the neglect a certain component of the spin we had to have already solved for the energies which already would have *involved that spin component*, i.e. using eq. (5.25) with eq. (5.21) as the integrand. Although intuitively easy this is logically invalid and probably glosses over interesting questions.

(2) The precessing solutions found in the homogeneous field of eq. (5.16) were rigorously obtained but then subjectively applied to a completely different problem

- one involving the field of eq. (5.6). This assumes that the interaction of these phenomena - the uniform and non-uniform parts of the field - is linear and can be naively superposed. However, it will be seen in Chapter 7 that solving only the non-uniform part is not trivial. This suggests that more than just a linear interaction is occurring.

(3) Finally, just because the expectation value time-averages to zero doesn't mean the *measured* value *is* zero or even that it is close to zero! According to the interpretation discussed in section 3.2.2 this only means that the average of several measurements all performed on identical systems will be zero but any one could be arbitrarily large. This says nothing about one measurement in particular.⁸

From all these considerations it is obvious that at best the precession argument that is traditionally invoked in thematic accounts of the SGE disguises several interesting questions and at worst is completely invalid and inaccurate.

5.5.3 Precession and the Uncertainty Principle

If there is a possibility that the precession argument is misapplied in the standard interpretation of the SGE then there is also a possibility of other misinterpretations as well. One of these has to do with the uncertainty principle.

We saw in section 3.2.4 that for any two non-commuting operators \hat{A} and \hat{B} such that

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = \hat{C} \quad (5.29)$$

in the quantum formalism there is a corresponding uncertainty relationship

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{|\langle\hat{C}\rangle|}{2} \quad (5.30)$$

which is typically interpreted as a constraint on the physical process of measurement. That is, as \hat{A} is measured to a given degree of precision the quantum state of the system being measured is altered in such a way as to limit the precision with which \hat{B} can be simultaneously measured.

⁸Exactly how much it says about a collection of many measurement is dictated by eq. (3.30).

Because the three components of the spin operator $\hat{\mathbf{S}}$ do not mutually commute they satisfy an uncertainty relationship as well

$$\Delta\hat{S}_i\Delta\hat{S}_j \geq i\epsilon_{ijk}\hat{S}_k. \quad (5.31)$$

We interpret this to mean that we cannot simultaneously measure two components of the spin to an arbitrary degree of accuracy. It is precisely the phenomenon of precession that allows this interpretation.

If for some reason precession could not be invoked as a valid occurrence then in the SGE our classical intuition would lead us to believe that the particle would arrive at the detection screen purely due to spin forces. If the particle were found at a 45° angle from the location of the localized y-directed beam in the field then we would assume the particle felt an equal force in both the x and z-directions. The formalism tells us that these forces, and thus deflections, arise in proportion to the spin component in the corresponding direction. Therefore, we would classically interpret such a result as a simultaneous measurement of both the x and z-components of the magnetic moment.

In a quantum context the same arguments apply but with more at stake. It is the relative strength of one direction to the other that justifies the presence of precession and it is this precession that gives the “washing out,” or ambiguity, of the transverse spin components necessary to the interpretation of the SGE in terms of the uncertainty principle. From this it seems that if the preferred direction were removed for the Stern-Gerlach measurement of a *single particle*, precession would also be removed and there would be nothing to rescue us from simultaneously measuring, or assigning definite values to, two orthogonal spin components of a single particle. Thus, it is possible that our present understanding of the uncertainty principle only follows from our choice of field and not from the nature of the particles themselves.

We accordingly will use the problems outlined here as a motivation and guide to the work and questions of the next chapter. In addition to the broad problems outlined so far there are many more smaller and more specific problems that arise in discussions of the SGE. These will be brought up in the appropriate places in the

next chapter as we investigate the nature of the SGE in several contexts and in more depth. This offers us several interesting possibilities for not only gaining a deeper understanding of the SGE but, more generally, of our interpretations of physics.

Chapter 6

The Proposal

The SGE is widely thought to have a description that is not only conceptually very clean with relatively few mathematical technicalities but that clearly demonstrates some fundamental characteristics of quantum behavior. Yet despite its clear break from classical expectations as we saw in the previous chapter our justifications for the SGE description have yet to break free of classical traditions. Consequently, there are few, if any, fully quantum descriptions that do not make reference to outdated notions.¹

As an example of this, we see that in order to motivate the use of a magnetic field which clearly violates Maxwell's equations many authors cite and/or derive the *classical* phenomenon of precession. This causes an averaging away of spin components transverse to the magnetic field so that the transverse behavior can be ignored from the beginning. However, as we saw in section 5.5 this is an *ad hoc* assumption and has not been shown to easily follow from rigorous solutions. We are therefore unaware of its implications. In addition, while this approach may be sufficient classically when applied to quantum mechanical descriptions of nature only *expectation values* can be expected to precess. Invoking this argument then in reference to the SGE blurs the distinction between classical and quantum conceptual systems which, because of their stark differences and the necessity of clear and consistent conceptual representations for rational communication (see section 2.1), leads to further confusion. If the quantum formalism is complete it should provide us with an appropriate

¹[28] and [29] make reference to this same thought so work in this area is mounting.

basis for describing the SGE.² Not only will this demonstrate consistency but it will allow us to uncover questions that have previously remained unasked merely because of self-enforced conceptual boundaries.

6.1 Positivist and Realist Representations

The formulation of a specific question that attempts to reveal the true nature and applicability of the SGE with its precession arguments can be motivated by contrasting two opposing conceptual, or representational, systems. We will call these the orthodox and realist views [1], [8].

The positivist position, which most closely resembles the orthodox or mainstream position, has become known as the Copenhagen interpretation. It states that prior to a measurement of a particular quantum property that property did not have a well defined value. It is the act of measurement that *compels* the particle to assume a definite value. In terms of a spin measurement via a Stern-Gerlach apparatus spin is not considered definite, i.e. it is not considered to take on *one* orientation or *one* magnitude, until the particle strikes the detection plate.³ Prior to this the most that can be said of the particle’s “spin” is that it was in a superposition of spin “up” and “down” states. In terms of position the particle was in a superposition of deflecting “up” and deflecting “down”.

In the realist opinion, these properties exist independent of the measurement process but are affected in complicated ways by it. In other words, the *existence* of these properties is independent but their specific *value* is not.⁴ As applied to spin, there is always a spin property to a particle, perhaps even zero, though its particular

²Because we have spoken of the SGE as an axiom of modern physics there could be an interesting discussion here on the applicability of Gödel’s Incompleteness Theorem.

³This statement emphasizes clarity at the expense of rightness. Exactly *when* measurement occurs is an open question contributing to the quantum problem of measurement summarized in section 3.2.3. Does it occur upon collision of particle with plate, with the gaze of a conscious observer, or somewhere in between (see [11] and [12])? In an ironic twist of quantum fate because we cannot measure measurement it seems we will never quite have a definite answer.

⁴The specific identification of the content of the realist and positivist positions is difficult because there is so little consensus as to particulars. We should point out that there are intermediary positions in which existence is still independent of measurement but completely unknowable due to the effects of the measurement process. Philosophically this might correspond to Kant’s discussion of noumena and phenomena. As stated here realism sounds very Randian. See [4].

orientation and magnitude are changed in ways that are heavily dependent upon the entire experimental arrangement, i.e. the distance to the screen, the orientation of the field, etc. [12]

Because in the positivist view the spin is not said to be an element of reality until measurement occurs it seems inconsistent to invoke the precession argument to justify the neglect of part of the field. Its neglect must be attributed to something other than the property of spin. The realist has no such problem. In this sense, this could serve as a crucial experiment for very particular views of realism and positivism.

6.1.1 The Changing Role of Precession

Because developing a deeper understanding of the SGE depends critically on the precession argument we must ascertain the true effect of precession, its validity, and its range of applicability. This can be done by considering its interaction with the spin or magnetic moment and its dependence on the homogeneous field component.

Is there any reason to apply this homogeneous field component? Its only purpose classically seems to be to induce precession about the direction of interest so that only that components in that direction will be clearly observed. As mentioned in section 5.5.3 the question becomes more interesting when we consider its implications in the quantum picture.

Although it seems it was introduced historically only to label the direction of interest it seems it has been preserved through the quantum revolution because our conceptual picture, the prevailing positivist view, requires it. Put differently, whereas in the old paradigm it was *justified* by our desire to measure only a single component of the magnetic moment, it has now become the *justification* for our belief that measuring only one component is possible, via the uncertainty principle.

This demonstrates the little noticed effect of paradigm on experimental practice. [2] explains that

consciously or not, the decision to employ a particular piece of apparatus and to use it in a particular way [as with the homogeneous field

component] carries an assumption that only certain sorts of circumstances will arise. (p. 59)

In addition to the usual theoretical expectations these instrumental expectations “have often played a decisive role in scientific development.”

6.2 Proposal: The Inhomogeneous Stern-Gerlach Effect

The most direct way that we propose to discover the true nature and effect of the precession argument on not only the measurement of spin but on our theoretical description and general interpretation of measurement as well is to theoretically remove the homogeneous field component which is the element that selects a universally preferred direction and which is the precession inducing agent. More explicitly, we propose to solve the SGE with a beam travelling in the y -direction by replacing the typically studied field configuration

$$\mathbf{B} = (B_0 + bz)\hat{z} \tag{6.1}$$

with the its more natural choice

$$\mathbf{B} = -bx\hat{x} + (B_0 + bz)\hat{z}. \tag{6.2}$$

We have included the constant component B_0 only so that the phenomenon of precession can be *explicitly shown* as opposed to its usual imposition. We can either set $B_0 = 0$ in order to solve the inhomogeneous SGE (ISGE) or leave it as a non-zero constant to test the known Stern-Gerlach limit.

Note that either choice is physically consistent with the requirement that

$$\nabla \cdot \mathbf{B} = 0. \tag{6.3}$$

The field does however blow up linearly far away from the origin.

6.2.1 The Experimental Arrangement

If we consider the physical realization of this field we can perhaps avoid this difficulty. The field eq. (6.1) may be considered an approximation to a configuration

of four parallel wires carrying current I located at the four corners of a rectangle of sides $2x_1$ and $2z_1$. With the origin of the x - z plane equidistant from each (figure ??) they may be described by the column vector

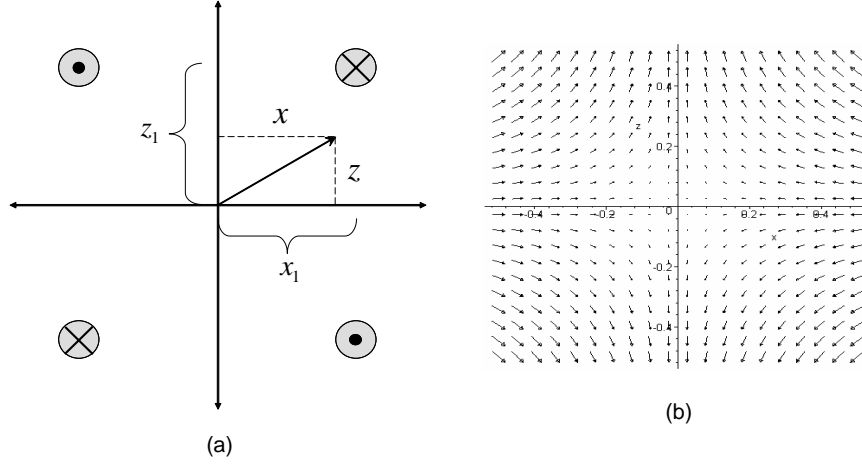


Figure 6.1: An arrangement of 4 long, parallel wires lying in the y -direction. Going from left to right, top to bottom the current flows out of the page (wire #1), into the page (wire #2), into of the page (wire #3), and out of the page (wire #4)

$$\mathbf{B} = \begin{pmatrix} B_x \\ B_z \end{pmatrix} = \tag{6.4}$$

$$= B_1 \begin{pmatrix} \frac{z_1-z}{(x_1+x)^2+(z_1-z)^2} - \frac{z_1-z}{(x_1-x)^2+(z_1-z)^2} + \frac{z_1+z}{(x_1+x)^2+(z_1+z)^2} - \frac{z_1+z}{(x_1-x)^2+(z_1+z)^2} \\ \frac{x_1+x}{(x_1+x)^2+(z_1-z)^2} + \frac{x_1-x}{(x_1-x)^2+(z_1-z)^2} - \frac{x_1+x}{(x_1+x)^2+(z_1+z)^2} - \frac{x_1-x}{(x_1-x)^2+(z_1+z)^2} \end{pmatrix} \tag{6.5}$$

where $B_1 = \frac{\mu_0 I}{2\pi}$ and the coordinate origin is naturally chosen at the field center.

At $(0,0)$ there is absolutely no field only an arbitrarily large gradient b . For us it would be ideal to send a point beam along the y -direction. This is however impossible. The beam must have some non-zero width. However, if we merely restrict ourselves to behavior *near* the origin we can drop all orders of x and z that are second order or greater with little consequence. For simplicity we will also take $x_1 = z_1$, i.e. the wires arranged in a square. Doing this arrives at precisely the field of eq. (6.1). In our search for solutions we must remember that any solutions we find are only valid for small x and z .

We can also give a form for the vector potential that satisfies this approximate field. It is

$$\mathbf{A} = B_0 xz \hat{y}. \quad (6.6)$$

However, even including minimal coupling as we have our choice to use only neutral particles will make the vector potential unimportant.

Sending a beam of neutral spin-1/2 particles, i.e. neutral silver atoms, along the y-axis in such a field will exhibit the ISGE.

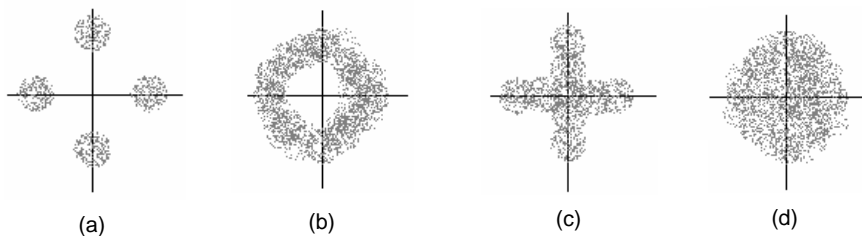


Figure 6.2: (a) The beam splits in both x and z -directions. (b) The beam splits in the radial direction only. (c) The beam splits into two angular direction. (d) The beam does not split but blurs in all directions. Some blurring will arise from collimation so in this case it would be important to note whether the observed blurring was attributable to spin separations.

6.2.2 Possible Outcomes

In the case that the $B_0 = 0$ choice is made what beam trace can we expect? Fig. 6.2 shows four possibilities. If we choose the experimental conditions, i.e. the time of interaction nT and the strength of the field gradient b , so as to invalidate precession arguments any result would be instructive. Moreover, because of the absence of precession, if it is run for only a *single* particle its deflection and detection *anywhere* on the plate would naively define 2-components of the “force” or, consequently, 2-components of the spin. For reasons discussed in section 5.5.3 this is of extreme interest.

Chapter 7

A More Complete Study of the Stern-Gerlach Effect

In accordance with the proposal of the previous chapter we seek here to find solutions and insights into the Inhomogeneous Stern-Gerlach Effect (ISGE). Because of the difficulty and nature of the task we will represent this problem in using several solution methods, formalisms, and pictures in an attempt to fully understand this effect in a quantum context along with its theoretical and philosophical implications.

7.1 Matrix Representation in a Moving Frame

In order to gain an initial familiarity with the full SGE, and in particular the ISGE, we can follow the method used in Chapter 3 only with the slightly modified field eq. (5.6). We begin with a Hamiltonian operator describing the ideally impulsive interaction of duration T as

$$\hat{H}(t) = \begin{cases} 0 & \text{for } t < 0 \\ -\frac{e}{m} \frac{\hbar}{2} \begin{pmatrix} (B_0 + bz) & -bx \\ -bx & -(B_0 + bz) \end{pmatrix} & \text{for } 0 \leq t \leq T . \\ 0 & \text{for } t > T \end{cases} \quad (7.1)$$

7.1.1 The Assumptions

This is a special representation of the Stern-Gerlach Hamiltonian. As was stated, but not discussed, in Chapter 3 this considers the dynamics from the frame of the particles in the beam which we take to be inertial. As this assigns the definite value of zero to p_x and p_z it correspondingly limits the possible definition of the position

according to eq. (3.31). The incoming beam is therefore a plane wave travelling in the y -direction with infinite extent in the x and z -directions. As it was shown in Chapter 3 the y -behavior can be separated off without any assumption on p_y .

For completeness we should also consider the effects of the idealization involving the impulsive field. In reality no field can be turned on infinitely quick or confined perfectly to a given region of space without some variation. In our case, in the rest frame of the beam particles, this would be manifest as a time-varying \mathbf{B} -field, which via Maxwell's equations sources electric fields. This could cause some complicated effects as the particle enters and exits the field. However, a numerical treatment of the ISGE involving charged particles using a field with a non-zero "turn-on" and "turn-off" time showed no significant differences from the idealized case [30]. Because of this and because we have chosen to work only with neutral particles we think that the idealization of an impulsive field is adequate.

7.1.2 The Eigenstates

The energies can be found in the same manner as in section 3.2.5 with a similar result. We get

$$E_{\uparrow\downarrow} = \mp \frac{e\hbar}{2m} \sqrt{(bx)^2 + (bz + B_0)^2}. \quad (7.2)$$

where the $\uparrow\downarrow$ refer two possibly resultant states *as represented in the z -basis*. We should also point out the apparent spatial dependence of the energies. This was also present in the treatment in Chapter 3. In order to make sense out of these spatially dependent eigenvalues the factors of x and z in eq. (7.2) should be thought of as parameters and not as spatial coordinates. If the beam were localized enough they might refer to the coordinates of the peak of the beam packet in the xz -plane. But for a beam that is extended compared to the region of interest as we have, (x, z) could be thought of as the initial position of the particles within the beam as in a realist, or pilot-wave-type visualization (See section 7.12.1 for further discussion). Either way, they should be thought of as parameters and not variables.

Although we will not need them we point out that the eigenvalues eq. (7.2) correspond respectively to the non-normalized eigenstates

$$|\uparrow\downarrow\rangle = \begin{pmatrix} 1 \\ -\frac{p_{\uparrow\downarrow}^2}{\epsilon\hbar bx} + \frac{B_0}{bx} + \frac{z}{x} \end{pmatrix} \quad (7.3)$$

with $p_{\uparrow\downarrow} = 2mE_{\uparrow\downarrow}$.

For $t < 0$ the field is off so we have a free particle at rest since $\hat{H} = 0$ ($E = 0$). At $t = 0$ the field is switched on and as we found earlier the states are describable as

$$\Psi = \psi_{\uparrow}\chi_{\uparrow}e^{i\frac{e}{2m}\sqrt{(bx)^2+(bz+B_0)^2}t} + \psi_{\downarrow}\chi_{\downarrow}e^{-i\frac{e}{2m}\sqrt{(bx)^2+(bz+B_0)^2}t}. \quad (7.4)$$

This persists until $t = T$ after which time the particle is again unaffected so the final states can be found by merely evaluating eq. (7.4) at $t = T$.

7.1.3 Demonstrating Precession

If we take the limit that ϵ is small where $\epsilon = b/B_0$ such that $|B_0| \gg |br|$ for all relevant values of $r = \sqrt{x^2 + z^2}$ we can demonstrate the effect of the precession argument but without reference to the classical or quantum concept of precession.

If we expand the squares in the phase ϑ of eq. (7.4) we can write

$$\vartheta = \pm i\frac{e}{2m}\sqrt{B_0^2 + 2B_0bz + b^2r^2}t. \quad (7.5)$$

Since we take B_0 to be large let us factor it out of the radical

$$\vartheta = \pm i\frac{eB_0}{2m}\sqrt{1 + \frac{2bz}{B_0} + \left(\frac{br}{B_0}\right)^2}t \quad (7.6)$$

Expanding the radical for small ϵ we get

$$\sqrt{1 + 2\epsilon z + \epsilon^2 r^2} = 1 + \epsilon z + \frac{\epsilon^2}{2}x^2 + \dots \quad (7.7)$$

So to first order the phase is

$$\vartheta = \pm i\frac{e}{2m}(B_0 + bz)t \quad (7.8)$$

which is exactly the phase of the solutions we found in Chapter 3 (see eq. (3.36)). This is a purely mathematical demonstration, within a few assumptions, of what is happening in the standard derivations of the SGE. It shows that despite the logical and

mathematical heuristics that often go into making a thematic discussion of the SGE intuitive there is a more mathematical and *clearly demonstrable* reason the physical picture works. Whether precession accounts for this is a matter of interpretation.

7.1.4 The Inhomogeneous Stern-Gerlach Effect

Now that we can clearly see that B_0 justifies the precession picture and the neglect of the x-directed field we can let $B_0 \rightarrow 0$ to solve the ISGE. This should clarify the role of precession even further.

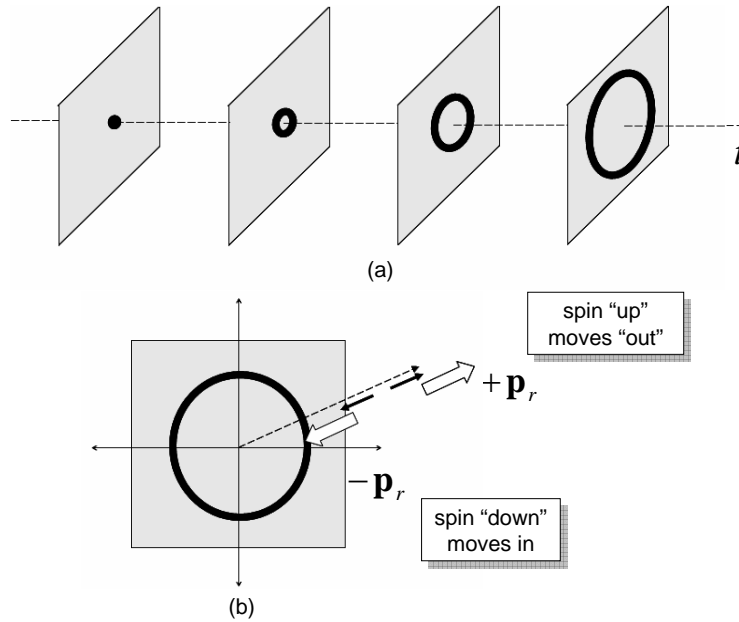


Figure 7.1: As time progresses the spins separate radially “up” or “down” depending on their spin orientation relative to the radial direction.

Returning to eq. (7.4) but with $B_0 = 0$ we can write the final inhomogeneous Stern-Gerlach states

$$\Psi = \psi_{\uparrow} \chi_{\uparrow} e^{\frac{i}{\hbar} \left(\frac{ebT\hbar}{2m} \right) r} + \psi_{\downarrow} \chi_{\downarrow} e^{-\frac{i}{\hbar} \left(\frac{ebT\hbar}{2m} \right) r}. \quad (7.9)$$

Using the same arguments as in section 3.2.5 we may interpret this as two circular plane waves travelling either radially outward or inward with a momentum

$$p_r = \pm \frac{ebT\hbar}{2m} \quad (7.10)$$

depending on its spin's projection along the z-axis. Relative to this axis the spin "ups" move radially outward while the spin "downs" move towards and through the center of the field.

7.2 Position and Momentum Representations

We can now attempt a more rigorous solution that involves fewer assumptions and compare. We begin this process with the time-dependent Schrödinger equation with operators in \mathbf{x} -space

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = -\frac{\hbar^2}{2m} \nabla^2 |\Psi\rangle + \hat{V} |\Psi\rangle, \quad (7.11)$$

where

$$|\Psi\rangle = \begin{pmatrix} \Psi_{\uparrow} \\ \Psi_{\downarrow} \end{pmatrix} \quad (7.12)$$

In our case the potential energy arises from the interaction of the magnetic moment $\hat{\boldsymbol{\mu}}$, or spin $\hat{\mathbf{S}}$, with the field $\hat{\mathbf{B}}$ as given in eq. (5.6). That is,

$$\hat{V} = -\hat{\boldsymbol{\mu}} \cdot \hat{\mathbf{B}}. \quad (7.13)$$

If we use the previous definitions of $\hat{\boldsymbol{\mu}}$ in terms of the Pauli spin matrices $\hat{\sigma}_j$ we have

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = -\frac{\hbar^2}{2m} \nabla^2 |\Psi\rangle - \frac{e\hbar}{m} \frac{\hat{\boldsymbol{\sigma}}}{2} \cdot \hat{\mathbf{B}} |\Psi\rangle. \quad (7.14)$$

We can easily separate off both the time and y-dependence using the same procedure as in section 3.2. Assuming $|\Psi\rangle = T(t)Y(y)|\psi(x, z)\rangle = TY|\psi\rangle$ we get the three equations

$$T(t) = T(0)e^{-iEt/\hbar} \quad (7.15)$$

$$Y(y) = Y(0)e^{ik_y y} \quad (7.16)$$

$$k^2 |\psi\rangle = -\nabla^2 |\psi\rangle - \frac{e}{\hbar} \hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{B}} |\psi\rangle, \quad (7.17)$$

where E and k_y are separation constants and $k^2 = 2m(E - k_y^2)/\hbar^2$. Also,

$$|\psi\rangle = \psi_{\uparrow}\chi_{\uparrow} + \psi_{\downarrow}\chi_{\downarrow} = \begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow} \end{pmatrix}. \quad (7.18)$$

Using the standard representation for the Pauli matrices $\hat{\sigma}_j$ we can write the matrix equation for $|\psi\rangle$ as two, coupled differential equations for the spin “up” and “down” components¹

$$k^2\psi_{\uparrow\downarrow} = -\nabla^2\psi_{\uparrow\downarrow} - \frac{eb}{\hbar} \left[x\psi_{\downarrow\uparrow} \mp z\psi_{\uparrow\downarrow} \right] \mp \frac{eB_0}{\hbar}\psi_{\uparrow\downarrow}. \quad (7.19)$$

This is actually two equations: one for ψ_{\uparrow} and one for ψ_{\downarrow} . Here and throughout the remainder of this work this compact notation is used in which the top (bottom) signs in \mp correspond to the first (second) subscript of $\psi_{\uparrow\downarrow}$.

In order to arrive at analytic solutions of this equation the “up” and “down” components must be decoupled. To do this would require applying the laplacian operator, ∇^2 , to both equations. Although this decouples the “up” and “down” behavior it also yields two fourth order partial differential equations (PDEs). This effectively prohibits us from using familiar differential equation solution techniques because they are typically only formulated for second order equations.

We conclude that although \mathbf{x} -space is an intuitive space to work in this context it requires techniques that we do not have. We can then transform eq. (7.19) into an “orthogonal” representation in hopes that the tradeoff between intuition and technical detail will be in our favor.

As it turns out, this is precisely what happens when mapping these equations to \mathbf{p} -space. When making this change from \mathbf{x} to \mathbf{p} -space, via a Fourier transformation, we make the following changes

$$x_j \rightarrow i\hbar \frac{\partial}{\partial p_j} \quad (7.20)$$

$$-i\hbar \frac{\partial}{\partial x_j} \rightarrow p_j, \quad (7.21)$$

¹Recall that “up” and “down” are in quotes only because they are terms relative to the basis by which we represented their spin operators. In our case, “up” and “down” in z .

that is, the coordinates become operators and the operators become coordinates. We also realize that

$$\psi_{\uparrow\downarrow}(x, z) \rightarrow \phi_{\uparrow\downarrow}(p_x, p_z). \quad (7.22)$$

Making this change eq. (7.19) becomes

$$k^2\phi_{\uparrow\downarrow} = -(p_x^2 + p_z^2)\phi_{\uparrow\downarrow} -ieb \left[\frac{\partial}{\partial p_x}\phi_{\uparrow\downarrow} \mp \frac{\partial}{\partial p_z}\phi_{\uparrow\downarrow} \right] \mp \frac{eB_0}{\hbar}\phi_{\uparrow\downarrow}, \quad (7.23)$$

in the momentum representation.

For convenience we write this in dimensionless form

$$\frac{\partial}{\partial p_x}\phi_{\uparrow\downarrow} \mp \frac{\partial}{\partial p_z}\phi_{\uparrow\downarrow} = -i \left[\xi - \alpha(p_x^2 + p_z^2) \pm \beta \right] \phi_{\uparrow\downarrow} \quad (7.24)$$

where τ and Δ are a characteristic time and length respectively. Also, $\xi \equiv k^2\hbar/e\Delta b$ is a unitless energy, $\alpha \equiv \hbar/e\Delta^3 b$, and $\beta \equiv B_0/\Delta b$ is the ratio of the homogeneous to the inhomogeneous field. Note also that the p_j are now dimensionless momentum variables.

The utility of this particular representation is that the derivative properties of \mathbf{x} -space are replaced with algebraic properties in \mathbf{p} -space. Thus, the *fourth* order PDE with *quadratic* terms in x and z that would arise from decoupling in \mathbf{x} -space now becomes much easier to decouple - a *second* order PDE that is now *quartic* in the coordinates p_x and p_z . Applying well known solution methods for PDEs therefore becomes much much more straightforward.

7.3 Rotated and Unrotated Representations: Decoupling

As it turns out, in the present form the decoupling process is quite messy. We can however rotate the basis in which they are represented in the complex plane and simplify the process.

It is useful to note that this is the same approach that we used in section 3.2.2 in which we rotated the coordinate system with which we described a function in order to simplify it algebraically except that our functions are complex. This rotation can take place in the complex plane.

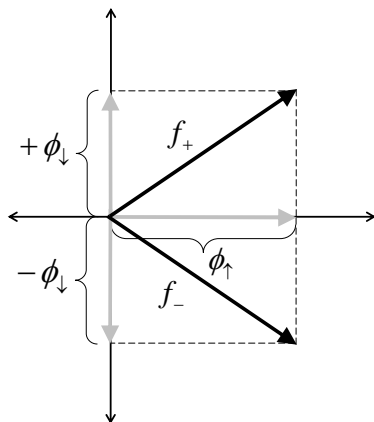


Figure 7.2: f_{\pm} can be thought of as a vector in the imaginary plane.

For simplicity, instead of dealing with the functions $\phi_{\uparrow\downarrow}$ we now choose to deal with the functions f_{\pm} where

$$f_{\pm} = \phi_{\uparrow} \pm i\phi_{\downarrow}. \quad (7.25)$$

Because the function f_{\pm} is a linear combination of the $\phi_{\uparrow\downarrow}$ it can be thought of as a simple 2-dimensional complex vector represented in the 2-dimensional ϕ -basis.²

Rewriting eq. (7.24) now we get

$$\hat{\mathcal{L}}_{\pm} f_{\mp} = \pm \left[\alpha(p_x^2 + p_z^2) - \xi \right] f_{\pm} \quad (7.26)$$

where $\hat{\mathcal{L}}_{\pm}$ is an operator of the form

$$\hat{\mathcal{L}}_{\pm} \equiv \frac{\partial}{\partial p_x} \pm \frac{\partial}{\partial p_z} \pm \beta \quad (7.27)$$

Keep in mind that these equations are actually a set of two equations written in a compact form.

²It is also interesting to note the similarity of f_{\pm} to right and left circularly polarized light. What this exactly means in the context of spin, for example, what sort of apparatus would filter these spin polarizations, is an interesting, and open, question.

With the introduction of f_{\pm} the decoupling of these equations is simpler. By applying $\hat{\mathcal{L}}_{\mp}$ to eq. (7.26) we can decouple to get

$$\hat{\mathcal{L}}_{\pm}\hat{\mathcal{L}}_{\mp}f_{\pm} + A\hat{\mathcal{L}}_{\pm}\left(\frac{1}{A}\right)\hat{\mathcal{L}}_{\mp}f_{\pm} - A^2f_{\pm} = 0 \quad (7.28)$$

where A is the function

$$A(p_x, p_z) \equiv \alpha(p_x^2 + p_z^2) - \xi. \quad (7.29)$$

Eq. (7.28) is two decoupled second order PDEs.

7.4 Cartesian and Polar Representations: Separation

If the p_x -dependence can be separated from the p_z -dependence then this equation can be treated as two ODEs instead of two PDEs which is a great simplification. Unfortunately in the cartesian representation such separation is not possible.

However we have noted that by transforming our coordinate system from a cartesian to a polar form we can simplify the functional form of the function A . Instead of being a function of 2 variables with the polar substitutions

$$\rho^2 = p_x^2 + p_z^2 \quad (7.30)$$

$$\varphi = \arctan\left(\frac{p_z}{p_x}\right) \quad (7.31)$$

in momentum coordinates the function A becomes a 1-dimensional function. Due to this symmetry the equations simplify. In fact, in the limit that $\beta \rightarrow 0$, or when the homogeneity is removed, they become separable. In particular, the equations for f_{\pm} become

$$(\xi - \alpha\rho^2)(\rho^2 f_{\pm\rho\rho} + f_{\pm\varphi\varphi}) + \rho(\xi + \alpha\rho^2)f_{\pm\rho\mp} \mp 2i\alpha\rho^2 f_{\pm\varphi} + (\alpha^3\rho^8 - 3\alpha^2\xi\rho^6 + 3\alpha\xi^2\rho^4 - \xi^3\rho^2)f_{\pm} = 0, \quad (7.32)$$

where each occurrence of the ρ or φ in the subscripts denote a partial derivative with respect to the corresponding variable.

For purposes of greater economy we let

$$\rho \rightarrow +\sqrt{\frac{\xi}{\alpha}}\sqrt{\rho} \quad (7.33)$$

giving eq. (7.32) the form

$$(1 - \rho)(4\rho^2 f_{\pm\rho\rho} + f_{\pm\varphi\varphi}) + 4\rho f_{\pm\rho} \mp 2i\rho f_{\pm\varphi} + \frac{\xi^3}{\alpha}\rho(\rho^3 - 3\rho^2 + 3\rho - 1)f_{\pm} = 0. \quad (7.34)$$

Now we can see that

$$f_{\pm}(\rho, \varphi) = R_{\pm}(\rho)P_{\pm}(\varphi) \quad (7.35)$$

is a suitable separation *ansatz* if we *choose*

$$P_{\pm}(\varphi) = e^{in_{\pm}\varphi} \quad (7.36)$$

as the solution to the angular part. By the single-valuedness requirement we know n_{\pm} can take on only integer values, $n_{\pm} = \dots, -2, -1, 0, 1, 2, \dots$, etc. The radial momentum part then becomes

$$R_{\pm}'' + \frac{1}{\rho(1 - \rho)}R_{\pm}' + \frac{\xi^3[\rho^4 - 3\rho^3 + 3\rho^2 + \alpha n_{\pm}(n_{\pm} \pm 2)\rho - (1 + \alpha n_{\pm}^2)]}{4\alpha\rho^2(1 - \rho)}R_{\pm} = 0, \quad (7.37)$$

an ODE in standard form with primes representing total derivatives. It is interesting to note that R_{\pm} is real valued and that $P_{+}(\varphi)$ is unchanged from $P_{-}(\varphi)$ except for the particular integer n_{+} or n_{-} .

7.5 Series Solution Representation

One of the most familiar solution techniques for a second order ODE such as we have is the Frobenius method. We first consult Fuch's theorem for the applicability of this method.

7.5.1 Singularity Structure and Fuch's Theorem

Fuch's theorem states that

...we can always obtain at least one power-series solution, provided we are expanding about a point that is an ordinary point or at worst a regular singular point [31].

Expansion about any other type of point may yield a solution. This theorem dictates only when obtaining a solution is guaranteed.

Following the methods of [31] p. 516-517 it can be shown that eq. (7.37) has regular points at $\rho = 0, 1$ and an irregular singular point at $\rho = \infty$. Choosing to do a Frobenius expansion about $\rho = 0$ is then the most straightforward choice because it guarantees us a solution.

7.5.2 The Indicial Equation

Using a series form for R_{\pm}

$$R_{\pm}(\rho) = \sum_{j=0}^{\infty} a_j \rho^{\lambda+j} \quad (7.38)$$

we can arrive at the indicial equation

$$\lambda = \pm \frac{n_{\pm}}{2}. \quad (7.39)$$

We accordingly take the large of the two roots $\lambda = +n \pm /2$, which is a half-integer.

7.5.3 Recurrence Relation

With the roots of the indicial equation we obtain a 5-term recurrence relation indexed by j

$$a_j = \frac{a_{j-4} - 3a_{j-3} + 3a_{j-2} - (1 - \eta(\lambda^2 + \lambda - j^2 + j))a_{j-1}}{\eta(j^2 - \lambda)} \quad (7.40)$$

where

$$\eta \equiv \frac{4}{(\Delta k)^6 \alpha}. \quad (7.41)$$

Following a promising comment by [32] p. 532 we make every attempt to reduce the number of terms in this recurrence relation from 5 to 3, perhaps even 2.

7.5.4 Extracting Behavior

Often the structure of the recurrence relation can be simplified if by some informed guess the asymptotic behavior can be extracted. This method is often employed when solving the quantum simple harmonic oscillator as in [1] and [33].

Unfortunately employing the same method here in various ways yielded no simplified result. In fact, in many cases by extracting the behavior at any or all of the singular points *complicated* the structure of the recurrence relation.

This suggests that we have hit some “critical mass” or “local minimum” of mathematical sophistication beyond which the problem increases in complexity. This is typical of non-linear systems as it seems that a simple combination of more basic parts can’t be trivially assembled to describe the full behavior. Nevertheless, there may still exist some more obscure choice by which the recurrence relation may be simplified. This is suggested by [32].

7.5.5 Truncation

Another common method also used in [1] and [33] to solve the quantum simple harmonic oscillator involves the truncation of the series. If a set of conditions can be found so as to force an appropriate set of consecutive coefficients in the series to go to zero then, by the recurrence relation, all succeeding coefficients will be zero as well. This truncates the infinite series into a finite polynomial. In the case of the simple harmonic oscillator this choice also provides the quantization condition on the energies.

However, because eq. (7.40) is a 5-term recurrence relation there are 5 independent conditions that must be made to go to zero in order to ensure *all* succeeding terms vanish as well. It was found that these 5 conditions are inconsistent, that is, some parameters were required to have two mutually exclusive values.

Although this is not a definitive proof it is evidence that in studying the ISGE we should not expect a quantization similar to that of the quantum simple harmonic oscillator. This realization can serve as a guide as to which systems we can model our solution methods and intuition after. For example, scattering and not bound systems might be preferred.

7.5.6 Radius of Convergence

Despite Fuch's theorem there remains no obvious way of reducing our recurrence relation to two or three terms. In order to rigorously test the convergence of the series a more compact form for the recurrence relation is needed. However, from numerical testing of convergence and from the location of the regular singularity at $\rho = 1$ we conclude that the radius of convergence for the solutions to eq. (7.37) is $-1 < \rho < 1$.

This tells us that the solutions we seek, if we could find them, are not normalizable for $-\infty < p_x < \infty$ and $-\infty < p_z < \infty$. By Parseval's theorem [31] this implies that the solutions are also not normalizable for $-\infty < x < \infty$ and $-\infty < z < \infty$ in the \mathbf{x} -representation. This suggests the fact that whatever solutions we are looking for they are either unphysical or require some subtle normalization procedure as in the delta-function normalization of plane waves. This may either be a result of the nature of the ISGE or of our crude and approximate field eq. (5.6) which we noted is unphysical since it blows up as $r \rightarrow \pm\infty$.³

7.5.7 Near-Origin Approximation

In replacing the four wire field of eq. (6.4) with the approximate field of eq. (6.1) we limited ourselves only to a study of the system's behavior near the *spatial* origin. The solutions we have generated are therefore only valid in this region. It is reasonable to assume then that if in eq. (7.37) we keep terms only up to first order in x and z , so as to focus on solutions for small values of x and z , i.e. near the origin, that we are actually only excluding insignificant behavior.

There is one complication however that should be considered. While this approximation is straightforward in \mathbf{x} -space we are now operating in \mathbf{p} -space which introduces other subtleties. For example, in the transformation from the \mathbf{x} -representation to the \mathbf{p} -representation we know that

$$x_j^n \rightarrow i\hbar \frac{\partial}{\partial p_j^n} \quad (7.42)$$

³It is unknown why both Fuch's theorem and the comment in [32] have not been substantiated.

which can be seen with a Fourier transformation. So higher powers of x_j become higher order derivatives with respect to p_j . In other words, the behavior of the solutions near the origin in \mathbf{x} -space is encoded in the *curvature* of the solutions *everywhere* in \mathbf{p} -space. Thus, in the process of keeping only near-origin behavior we will drop all second order derivative terms from eq. (7.37). Although this seems to make the appropriate restriction the neglect of derivatives in an ODE typically changes the nature of the equation completely. Thus, the question remains open as to how crude this is as an approximation.

Perhaps the neglect of second order derivatives is no more crude than it was in the \mathbf{x} -space method used in Chapter 3. There we assumed we could derive a solution *in the rest frame of the particles*. Since a reasonable result was obtained there it didn't seem to be too restrictive although it was a technical dropping of a second order derivative operator ∇^2 . This effectively defined the particle momentum in x and z as exactly zero, which by the uncertainty relation eq. (3.31), completely blurs its position in the corresponding plane. Thus, it was equivalent to describing the beam as an infinite plane wave.

In the context we use it here the approximation is not so clearly understood. It is however mathematically equivalent so we have good reason to believe that it has a similar although much less familiar interpretation.

Having said this we make the approximation and our second order ODE of eq. (7.37) reduces to first order. This can be easily integrated. Combining again with the angular solutions with $n_+ = n$ and $n_- = m$, we get

$$\phi_{nm\uparrow\downarrow} = \rho^{1+\frac{\xi^3}{4\alpha}} \left[A_{nm} \rho^{\alpha n^2} e^{-\frac{\xi^3 \rho}{16\alpha} (\rho((\rho-2)^2+2)+4n\alpha(n+2))} e^{in\varphi} \right. \quad (7.43)$$

$$\left. \pm i B_{nm} \rho^{\alpha m^2} e^{-\frac{\xi^3 \rho}{16\alpha} (\rho((\rho-2)^2+2)+4m\alpha(m-2))} e^{im\varphi} \right] e^{i(k_y y - E/\hbar t)} e^{-iE_{\uparrow\downarrow} t/\hbar}. \quad (7.44)$$

These are the stationary state wave functions for the “up” and “down” components near the origin in \mathbf{p} -space for the ISGE. A general solution would be a linear combination of these.

The probability density (PD) of either the “up” or “down” components is found using

$$\text{PD} = \phi_{\uparrow\downarrow}^* \phi_{\uparrow\downarrow}. \quad (7.45)$$

Applying this to our solutions we get

$$\text{PD for } \uparrow\downarrow = \rho^{2+\frac{\xi^3}{2\alpha}} \left[|A_{nm}|^2 \rho^{\alpha n^2} e^{-\frac{\xi^3 \rho}{8\alpha} N} \right. \quad (7.46)$$

$$\pm i \rho^{\alpha(n^2+m^2)} e^{-\frac{\xi^3}{16\alpha}(N+M)} \quad (7.47)$$

$$\left(A_{nm}^* B_{nm} e^{-i(n-m)\varphi} - A_{nm} B_{nm}^* e^{i(n-m)\varphi} \right) \quad (7.48)$$

$$\left. - |B_{nm}|^2 \rho^{\alpha m^2} e^{-\frac{\xi^3}{8\alpha} M} \right] \quad (7.49)$$

where

$$N = \rho((\rho - 2)^2 + 2) + 4n\alpha(n + 2) \quad (7.50)$$

$$M = \rho((\rho - 2)^2 + 2) + 4m\alpha(m - 2) \quad (7.51)$$

and also A_{nm} and B_{nm} are constants of integration.

Realize that A_{nm} and B_{nm} relate to $R_{\pm}(0)$. This in turn relates to $f_{\pm}(0)$ which relates to *both* $\phi_{\uparrow\downarrow}(0)$. Furthermore, eq. (7.45) is a probability density for a given spin *in \mathbf{p} -space*. Thus, it is not trivial to correlate A_{nm} and B_{nm} to the initial spin states or give eq. (7.43) or eq. (7.45) a clear physical interpretation.

7.6 The Confluent Heun Equation

We could also take advantage of the fact that eq. (7.37) is strikingly similar to the Heun equation (see [34], [35])

$$y''(x) + \left[\frac{a_1}{x} + \frac{a_2}{x-1} + \frac{a_1+a_2}{x-x_0} + \frac{a_3}{x^2} + \frac{a_4}{(x-1)^2} + \frac{a_5}{(x-x_0)^2} \right] y(x) = 0. \quad (7.52)$$

This is in normal form. It has one regular singularity at each of four points $x = 0, 1, x_0, \infty$.

By conflating the singularities $x = x_0$ and $x = \infty$ we get the confluent Heun equation (CHE)

$$y''(x) + \left[\tilde{a}_0 + \frac{\tilde{a}_1}{x} + \frac{\tilde{a}_2}{x-1} + \frac{\tilde{a}_3}{x^2} + \frac{\tilde{a}_4}{(x-1)^2} \right] y(x) = 0 \quad (7.53)$$

which has the same singularity structure as eq. (7.37) at least in terms of number and location.

As it turns out this is not enough. Although the CHE has an irregular singularity at $x = \infty$, as does eq. (7.37), it is not of the same rank. In other words, the singularity at infinity for the CHE does not blow up as fast as the corresponding singularity in eq. (7.37). We have tried to extract at least a portion of the asymptotic behavior as discussed in [34] and [35] in an attempt to rectify these two singularities but we have not been successful. Thus, we have learned that a discrepancy in the *rank* of the singularities is enough to prohibit our use of the CHE in describing the ISGE.

7.7 Clifford Representation

There are many mathematical representations we could use to facilitate the solution process of the ISGE. As each one is designed with a certain end in mind each one has its particular strengths and limitations. We have already seen the problem of the ISGE treated using both matrices and differential calculus and encountered their special challenges as well as their advantages. Another mathematical space within which the quantum phenomena of spin is particularly interesting is the Clifford algebra Cl_3 , also known as the Pauli algebra.

In this algebra the Pauli matrices $\hat{\sigma}_j$ are not given a particular matrix representation but are treated in a representation-free manner. More particularly, we define a space with an associated product that preserves the algebraic properties of the $\hat{\sigma}_j$ but that does not require a specific matrix representation to manipulate them. In this space the *matrices* $\hat{\sigma}_j$ become *vectors* \mathbf{e}_j and can be taken as an orthonormal basis that spans the space. In this way the spin properties, represented by the algebraic properties of the \mathbf{e}_j , are associated with the physical 3-dimensional space spanned by the three \mathbf{e}_j in a natural way (see [36]). By avoiding matrix representations we can avoid some of the complications that arise and focus on the phenomenon of interest.

If we express eq. (7.14) in a frame moving along with the particles in the beam, as we did in section 6.2 we can neglect the kinetic energy terms. We then have

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = -\frac{e}{m}\frac{\hbar}{2}\hat{\sigma}_j B_j|\Psi\rangle \quad (7.54)$$

where there is an implied sum over j and B_j is just a scalar. Using the simple mappings

$$|\psi\rangle \rightarrow \psi = \psi^0 + \mathbf{i}\psi \quad (7.55)$$

$$\sigma_j|\psi\rangle \rightarrow \mathbf{e}_j\psi\mathbf{e}_3 \quad (7.56)$$

$$i|\psi\rangle \rightarrow \mathbf{i}\psi\mathbf{e}_3 \quad (7.57)$$

$$i\sigma_j|\psi\rangle \rightarrow \mathbf{i}\mathbf{e}_j\psi \quad (7.58)$$

that can be found in [36] we can express eq. (7.54) in Cl_3

$$\frac{d\psi}{dt} = \frac{e}{2m}\mathbf{i}\mathbf{B}\psi. \quad (7.59)$$

where ψ is a Clifford representation, or a cliffor, of a spinor in Cl_3 and \mathbf{i} is a cliffor with the same algebraic properties as the usual unit imaginary i but can additionally be interpreted as an oriented unit volume in Cl_3 . $\mathbf{i}\mathbf{B}=\mathbf{i}B_j\mathbf{e}_j$ is the magnetic bivector corresponding to a plane normal to \mathbf{e}_j . Because we have neglected translational terms in eq. (7.54) the spatial dependence of ψ is negligible and so the partial derivative has been replaced with a total derivative. This makes the solutions easier but surely hides important information.

The cliffor equation eq. (7.59) can be easily integrated to yield

$$\psi(t) = e^{\frac{e}{2m}\mathbf{i}\mathbf{B}t}\psi(0). \quad (7.60)$$

The particular form of \mathbf{B} has not been given yet. It has only been assumed that \mathbf{B} is constant in time. Substituting eq. (5.6) with $B_0 = 0$, ψ becomes

$$\psi(t) = e^{\frac{eb}{2m}(-x\mathbf{i}\mathbf{e}_1+z\mathbf{i}\mathbf{e}_3)t}\psi(0). \quad (7.61)$$

If we define a space dependent unit bivector A such that

$$A(x, z) \equiv \frac{-x\mathbf{i}\mathbf{e}_1 + z\mathbf{i}\mathbf{e}_3}{|-x\mathbf{i}\mathbf{e}_1 + z\mathbf{i}\mathbf{e}_3|} \quad (7.62)$$

then ψ clearly takes the form

$$\psi(t) = e^{(\omega t)A}\psi(0) \tag{7.63}$$

where $\omega = eb|A(x, z)|/2m$. In this form it is easily seen to be the clifford representation of rotations in the plane defined by the bivector $A(x, z)$ [36]. Therefore, this picture of the ISGE only emphasizes the local precession of the spin state ψ at a location dependent characteristic frequency ω . However, the identification of two distinct momenta entangled with spin is not as clear.

If we would have included the kinetic energy terms in eq. (7.54) we could have used other geometric techniques of Cl_3 . The solution method above demonstrates that the removal of the matrix level of representation in Cl_3 causes systems of coupled equations like eq. (7.19) to be replaced by a *single* equation in which *separation* of the variables becomes the issue. This could be a very desirable result depending on the availability and familiarity of either decoupling and separation techniques.

7.8 Green's Function Representation

Section 7.2-7.6 exhausted several possibilities in order to solve the second order PDE. In the end the only thing we could do to avoid diverging solutions was to drop the second derivative terms the \mathbf{p} -space equation. This made the series convergent but may have also inadvertently excluded other interesting behavior. By applying other methods we may be able to avoid this.

In section 7.5.5 we concluded that the Stern-Gerlach system is similar to scattering systems. In fact, we can consider it a special case of a scattering problem in which an incoming beam of particles undergoes a magnetic interaction via a magnetic potential and spins. Represented in this way the tools and methods of canonical quantum scattering theory become tools and methods easily adaptable to understanding the ISGE. For example, Green's functions, propagators, and the Born approximation may be applied to solve the ISGE. These methods are familiar in \mathbf{x} -space.

We can think of the time-independent Schrödinger equations for spin “up” and “down” as two Helmholtz equations sourced by the functions $g_{\uparrow\downarrow}(x, z)$

$$(\nabla^2 + k^2) \psi_{\uparrow\downarrow} = g_{\uparrow\downarrow} \quad (7.64)$$

where

$$g_{\uparrow\downarrow} = \frac{eb}{\hbar} (x\psi_{\uparrow\downarrow} \mp z\psi_{\uparrow\downarrow}). \quad (7.65)$$

Notice that $g_{\uparrow\downarrow}$ couples the equations.

In general, the sources $g_{\uparrow\downarrow}$ are extended functions over some region of space. The Green’s functions $G_{\uparrow\downarrow}(x, z; x', z')$ are defined as the spin “up” or “down” field component at point (x, z) produced by a unit point source located at (x', z') . $G_{\uparrow\downarrow}$ therefore satisfies

$$(\nabla^2 + k^2) G_{\uparrow\downarrow} = -\delta(x - x')\delta(z - z'). \quad (7.66)$$

[31] gives the solution to these equations, the 2-dimensional Helmholtz Green’s functions for unbounded space, as

$$G_{\uparrow\downarrow} = \frac{-1}{4k_x k_z} e^{ik_x(x-x')} e^{ik_z(z-z')} \quad (7.67)$$

identical for both spin “up” and “down” cases with $k^2 = 2mE/\hbar^2 = k_x^2 + k_z^2$.

According to the formal theory of Green’s functions we can construct the solutions to the whole field $\psi_{\uparrow\downarrow}$ in eq. (7.64) by treating the source $g_{\uparrow\downarrow}$ as a collection of point sources and summing over their individual field contributions $G_{\uparrow\downarrow}$. That is,

$$\psi_{\uparrow\downarrow}(x, z) = \int_{allspace} G_{\uparrow\downarrow}(x, z; x', z') g_{\uparrow\downarrow}(x', z') dx' dz'. \quad (7.68)$$

7.8.1 A Magnetic Field with Gaussian Fall Off

The advantage of approaching the problem from the perspective of these integral equations as opposed to the differential equations of previous sections is that it is much easier to include a magnetic field that has a realistic asymptotic behavior. We can make a slight modification to our field here that would not have been practical in our earlier approaches.

If we pick \mathbf{B} such that

$$\mathbf{B} = b(-x\hat{x} + z\hat{z})e^{-(x^2+z^2)/a}, \quad a > 0 \quad (7.69)$$

where a is some characteristic length scale, it will capture the inhomogeneous behavior of the four wire field near the origin but will also provide appropriate fall off at large distances $r \gg a$. This gaussian factor would have greatly complicated the ODE approach of section 7.2-7.6, especially in \mathbf{p} -space, but can be more easily used with integrals. Doing this will hopefully eliminate much of the problems with the ODE approach of the previous section which seemed to result from the unrealistic field configuration.

7.8.2 The Born-Approximation

In order to get around the fact that $g_{\uparrow\downarrow}$ in eq. (7.68) couples the two solutions together we must either decouple the equations, which will lead to two fourth-order operators, or we can use Born's iterative approximation method with a well known Helmholtz Green's function. Sacrificing rightness for clarity we choose the latter approach. in principle, it can easily be employed to obtain solutions up to any desired accuracy. Because the field now decays at large distances there is hope that only a few iterations will capture the essential behavior of the ISGE.

In this method the wave function $\psi_{\uparrow\downarrow}$ is seen as a sum of successively smaller corrections

$$\psi_{\uparrow\downarrow} = \psi_{\uparrow\downarrow}^{(0)} + \psi_{\uparrow\downarrow}^{(1)} + \psi_{\uparrow\downarrow}^{(2)} + \psi_{\uparrow\downarrow}^{(3)} \dots \quad (7.70)$$

Note here that the larger orders (numbers in parentheses) label successively smaller corrections. Each successive order is found by using eq. (7.68) with $g_{\uparrow\downarrow}$ approximated from the preceding order.⁴ So,

$$\psi_{\uparrow\downarrow}^{(n)}(x, z) = \int_{allspace} G_{\uparrow\downarrow}(x, z; x', z') g_{\uparrow\downarrow}^{(n-1)}(x', z') dx' dz' \quad (7.71)$$

⁴It is interesting to note that we also applied this iterative technique to the differential equation. We solved a homogeneous differential equation then substituted that solution back into the equation now with an approximated source. When this was iteratively done the solutions were found to blow up, likely for the same reasons other derivative methods did as well although when checked this approach worked for simpler cases.

where $\psi_{\uparrow\downarrow}^{(n-1)} = \psi_{\uparrow\downarrow}^{(0)} + \psi_{\uparrow\downarrow}^{(1)} + \dots + \psi_{\uparrow\downarrow}^{(n-2)}$ and

$$(\nabla^2 + k^2) \psi_{\uparrow\downarrow}^{(0)} = 0. \quad (7.72)$$

In this way, beginning with the homogeneous (zeroeth order) solutions to the Helmholtz equation - the free particle solutions

$$\psi_{\uparrow\downarrow}^{(0)}(x, z) = e^{ik_x x} e^{ik_z z} \quad (7.73)$$

- and proceeding through iteration, the solutions to the ISGE can be found to any desired degree of accuracy. While this can be done we will not pursue it here. We will instead comment later on a similar propagator approach in section 7.11.1.

7.9 Schrödinger and Heisenberg Representations

The final representation that we will discuss in technical detail has less to do with the spatial behavior of the solutions and focuses more on the dynamics. Early on, before we even moved into either \mathbf{x} or \mathbf{p} -space we assumed the states $|\Psi\rangle$ carried the time characteristics of the evolution of the system. This is the Schrödinger representation of quantum mechanics which is expressed in his equation for $|\Psi\rangle$

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle \quad (7.74)$$

given in section section 3.2. We could have associated the time development with the *operators* involved instead. Because states are only revealed to us through measurement - the action of an operator - this is really an indistinguishable and arbitrary choice. Such as choice constitutes the Heisenberg representation. This follows Heisenberg's equation

$$i\hbar \frac{\partial}{\partial t} \hat{A} = [\hat{A}, \hat{H}] + i\hbar \frac{\partial}{\partial t} \hat{A} \quad (7.75)$$

for an operator \hat{A} as also given in section 3.2.

There is another possibility. Instead of an all-or-nothing treatment we could choose to associate *some* of the time-dependence with the state and some with the operators. This is known as the Intermediate, or Dirac, picture [33]. This gives us much more freedom as we can choose from many different options exactly how to divide up the dynamics of the system. We will discuss two particular ways here.

7.9.1 A Mixed Picture

If our Hamiltonian for the ISGE is

$$\hat{H} = \hat{T} + \hat{V} \quad (7.76)$$

then we can choose to divide up the time-dependence in many ways. For example, we can treat our operators \hat{A} , including \hat{T} , in the Heisenberg picture as varying in time while treating \hat{V} using the Schrödinger picture with the states carrying the time-dependence. Thus we may separate our equation into two parts

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \hat{V} |\Psi\rangle \quad (7.77)$$

$$i\hbar \frac{\partial}{\partial t} \hat{A} = [\hat{A}, \hat{H}] + i\hbar \frac{\partial}{\partial t} \hat{A} \quad (7.78)$$

where, in the case of the ISGE,

$$\hat{A} = \hat{\mathbf{p}}^2, \hat{\boldsymbol{\sigma}}, \hat{\mathbf{x}}, \text{ etc.} \quad (7.79)$$

$$\text{and } \hat{V} = -\hat{\boldsymbol{\mu}} \cdot \hat{\mathbf{B}}. \quad (7.80)$$

Since we take neither T nor V to have *explicit* time-dependence we can drop the partial derivative terms and can evaluate the commutators in eq. (7.78) for the operators listed in eq. (7.79) to yield

$$\frac{d}{dt} \hat{\mu}_j = 0 \quad (7.81)$$

$$\frac{d}{dt} \hat{x}_j = \frac{\hat{p}_j}{m} = \hat{v}_j \quad (7.82)$$

$$\frac{d}{dt} \hat{p}_j = 0. \quad (7.83)$$

So $\hat{\mu}_j$ and \hat{p}_j are constant in time and $\hat{x}_j = \hat{v}_j t + \hat{x}(0)_j$.

Solving eq. (7.77) is very similar to what we did before in section 7.2 except that now there is no kinetic energy term. We have not just dropped it as was done in sections 7.1 and 7.7 but we have treated it elsewhere, namely in eq. (7.78) above. In this way the present approach is more rigorous than what we have done before. Nevertheless, we expect it to yield similar results. Following the same steps as in section 6.3 we arrive at

$$\frac{d}{dt} \psi_{\uparrow\downarrow} = -i \frac{eb}{2m} (x\psi_{\downarrow\uparrow} \mp z\psi_{\uparrow\downarrow}). \quad (7.84)$$

The equations are still coupled but because of our particular choice of representation there are some significant differences from our previous approaches. They are:

- (1) The equations are first order in time as opposed to second order in space.
- (2) Because there is only one time variable but two spatial variables these equations are ODEs not PDEs.
- (3) The equations are more general because the kinetic energy is taken into account but simpler because it is treated separately.

Due to (1) the equations can be decoupled to yield the two second order ODEs

$$x \frac{d^2}{dt^2} \psi_{\uparrow\downarrow} - v_x \frac{d}{dt} \psi_{\uparrow\downarrow} + \left[\left(\frac{eb}{2m} \right)^2 (x^3 + xz^2) \mp \frac{eb}{2m} (v_x z - x v_z) \right] \psi_{\uparrow\downarrow} = 0 \quad (7.85)$$

where x and z are functions of time in general (see eq. (7.82)).

In the limit that we take the incoming wave to be an infinite plane wave as we did before we can recover the same results. More specifically, for an infinite plane wave travelling in the y -direction v_x and v_z both have definite zero values. Making this assignment eq. (7.85) becomes

$$\frac{d^2}{dt^2} \psi_{\uparrow\downarrow} + \left(\frac{eb}{2m} \right)^2 (x_0^2 + z_0^2) \psi_{\uparrow\downarrow} = 0. \quad (7.86)$$

The x and z reduce to x_0 and z_0 respectively because $v_x = v_z = 0$.

The solutions to this equation are

$$\psi_{\uparrow\downarrow} = C_1 e^{\frac{i}{\hbar} \left(\frac{eb\hbar t}{2m} \right) r_0} + C_2 e^{-\frac{i}{\hbar} \left(\frac{eb\hbar t}{2m} \right) r_0}. \quad (7.87)$$

If the interaction lasts for time T then this can be interpreted in the same manner as in section 3.2.5. In fact, the momenta

$$p_r = \pm \frac{eb\hbar T}{2m} \quad (7.88)$$

is identical. However, in this approach we see *as a consequence of the derivation* and not as a matter of interpretation as before that the spatial dependence in the exponent is actually a parameter describing the *initial* position of the beam or particle in the field.

From this approach we can also more naturally attribute the constants C_1 and C_2 to the initial spin conditions in the z-basis. C_1 is the fraction of the initial spins that were in the spin “up” direction and C_2 is the fraction initially in the spin “down” direction as defined in the z-basis. For example, if the initial beam is perfectly polarized beam in $+z$, i.e. spin “up”, then $C_1 = 1$ and $C_2 = 0$

$$\psi_{\uparrow\downarrow} = e^{\frac{i}{\hbar}(\frac{eb\hbar t}{2m})r_0} \quad (7.89)$$

and we have a plane wave travelling radially outward the radial direction being defined as pointing from the origin to r_0 .

If the spins were initially polarized in the “down” direction then the plane wave would be travelling radially inward continuing on through the center.

We can also discuss orthogonal spins. Say the initial beam had initially passed through a Stern-Gerlach magnet such that the resulting polarization was in the x-direction. In this case $C_1 = 1/\sqrt{2}$ and $C_2 = 1/\sqrt{2}$. Thus

$$\psi_{\uparrow\downarrow} = \frac{1}{\sqrt{2}}e^{\frac{i}{\hbar}(\frac{eb\hbar t}{2m})r_0} + \frac{1}{\sqrt{2}}e^{-\frac{i}{\hbar}(\frac{eb\hbar t}{2m})r_0} \quad (7.90)$$

and we have half the beam separating in the $+r$ -direction and half in the $-r$ -direction just as we might expect based on the results of the traditional SGE.

Notice this approach either tells us nothing at all about the angular behavior of the solutions or it tells us that they are angularly symmetric. Compare this with eq. (7.36) of section 6.5.

If to any of the foregoing derivations for the ISGE a large field component $B_0\hat{z}$ were added it is presumed that the radial behavior would partially average leaving only the behavior of the selected direction thus recovering the traditional SGE limit.

7.9.2 The Heisenberg Picture

There are other choices within the Intermediate picture we could make in dividing up the time-dependence of the system. [28] chooses to assign all temporal behavior to the operators thus adopting a purely Heisenberg approach.

[28] also uses a field similar to eq. (6.1) but truncated to a finite region by a step function in order to derive both the standard SGE as well as the ISGE. Because

the method is very similar to ours above we only cite their results. They find for the expectation values of x and z

$$\langle x(t) \rangle = \langle x_0 \rangle + \frac{t}{m} \langle p_{x_0} \rangle + \frac{eb\hbar t}{2m^2v} \langle \sigma_{x_0} \rangle \quad (7.91)$$

$$\langle z(t) \rangle = \langle z_0 \rangle + \frac{t}{m} \langle p_{z_0} \rangle - \frac{eb\hbar t}{2m^2v} \langle \sigma_{z_0} \rangle \quad (7.92)$$

with the assumptions that the factor $eb\hbar/2m^2v$ is small enough to neglect the second order term and that the velocity v of the beam in the laboratory frame is constant.

They conclude that

...in fact only the *average* x deflection vanishes. A spin-up particle will be found to undergo an x displacement but with equal probabilities in the $+x$ and $-x$ directions. (p. 580)

This, they claim, can be more clearly seen from the rest of their paper which we will discuss later in section 6.14.1.

7.10 Comment on Precession in the Inhomogeneous Stern-Gerlach Effect

We have found some evidence that in the ISGE the particles will undergo a deflection in the positive radial direction for one sense of spin orientation and in the negative radial direction for the other. This radial direction is defined by the line connecting the particle's parameterically described initial position in the beam r_0 and the origin. This is well defined since $(0, 0)$ is the only point where $\mathbf{B}=0$.

7.10.1 The Inhomogeneous Stern-Gerlach Effect as a Local Stern-Gerlach Experiment

To see what exactly is going on at the field point r_0 let's consider our purely inhomogeneous magnetic field

$$\mathbf{B} = b(-x\hat{x} + z\hat{z}) \quad (7.93)$$

which was valid in a small region equidistant from the four parallel wires running in the y-direction (see figure ??). Let us suppose that in a coordinate system centered on the point $\mathbf{B}=0$ that our particle is initially at the point

$$\mathbf{r} = r_0 \hat{r} = x_0 \hat{x} + z_0 \hat{z}. \quad (7.94)$$

The field at this point is

$$\mathbf{B}(x_0, z_0) = b(-x_0 \hat{x} + z_0 \hat{z}). \quad (7.95)$$

If we add and subtract this field (eq. (7.95)) to the field everywhere (eq. (7.93)) then although there is no change we can write the general field as

$$\mathbf{B} = b \left[(-x + x_0) \hat{x} + (z + z_0) \hat{z} \right] - \mathbf{B}(x_0, z_0). \quad (7.96)$$

Now doing a coordinate transformation such that

$$\tilde{x} = x - x_0 \text{ and } \tilde{z} = z + z_0 \quad (7.97)$$

we have

$$\mathbf{B} = b(-\tilde{x} \hat{x} + \tilde{z} \hat{z}) + \mathbf{A} \quad (7.98)$$

where

$$\mathbf{A} = -\mathbf{B}(x_0, z_0) = -b(x_0 \hat{x} + z_0 \hat{z}) = br_0 \hat{r} \quad (7.99)$$

is a *local* homogeneity along the direction pointing from $(0,0)$ to (x_0, z_0) in the (x, z) coordinates.

We can then interpret the ISGE as a *local* SGE in which there is a local uniform field component about which precession can be thought to occur as well as a non-uniform component which causes the particles *in that region* to separate according to their spin orientations. However, at any given point

$$|\mathbf{B}_{homogeneous}| = br_0 = b\sqrt{x_0^2 + z_0^2} \quad (7.100)$$

and

$$|\mathbf{B}_{inhomogeneous}| = b\sqrt{x_0^2 + z_0^2 + x^2 + z^2 - 2xx_0 + 2zz_0} \quad (7.101)$$

so there is no clear reason to believe that the validating condition eq. (5.11) is met for precession argument.

7.10.2 Making Precession Insignificant: A Semi-Classical Argument

As we have said, one can trivialize the occurrence of precession in another way. If the experiment is constructed in such a way that the apparent duration of the field to the particle is short compared to the precession period then the averaging procedure exemplified in eq. (5.25) does not go to zero (see section 5.5). The precession argument becomes invalid.

As we have seen, in the ISGE the incident beam seems to spread in the radial direction with momentum

$$\mathbf{p} = \pm \frac{eb\hbar T}{2m} \hat{r}. \quad (7.102)$$

So as the field gradient b or the time of interaction T increase, the “up” and “down” components of the incident wave are able to spread further apart.

Let us assume that, as is the case with silver atoms, the particles are massive enough that it is consistent to discuss them in terms of trajectories on their way to the detecting plate. This justifies a semi-classical treatment of the ISGE in which the particles are thought to have classical-like trajectories but with a “spread” factor appended. That is,

$$p_j \rightarrow p_j + \Delta p_j \quad (7.103)$$

where $|\Delta p_j$ follows the uncertainty relation eq. (3.31).

Following this we can say that the particles have distinguished their trajectories according to their spin “up” and spin “down” components in the direction of the radial vector \mathbf{r}_0 by assuming different \mathbf{p} 's or propagation directions.

If the radial distance travelled is R during the time of flight τ at a constant speed v then we may write

$$p_r = \frac{eb\hbar T}{2m} = mv = m \left(\frac{R}{\tau} \right) \quad (7.104)$$

so

$$R = \frac{eb\hbar T \tau}{2m^2}.^5 \quad (7.105)$$

⁵Notice the similarities between this and the small factor in eq. (7.91) and eq. (7.92).

However, even if we neglect the splitting that occurs from the magnetic field some blurring of the beam will occur due to diffraction effects of the beam collimation, i.e. effects from the Δp_j in eq. (7.103). In a quantum context this exemplifies the uncertainty principle. If we confine the beam to a given region of space through collimation we necessarily broaden the spread of momenta in that direction. So let us confine all particles in the beam to a circular beam region of radius Δr_0 . The spread of radial momentum obeys

$$\Delta r_0 \Delta p'_{r_0} \geq \frac{\hbar}{2}, \quad (7.106)$$

the uncertainty principle. Note that the primes refer to dynamics that result from *collimation* whereas the unprimed variables refer to the dynamics caused by the interaction of the particle with the *field*.

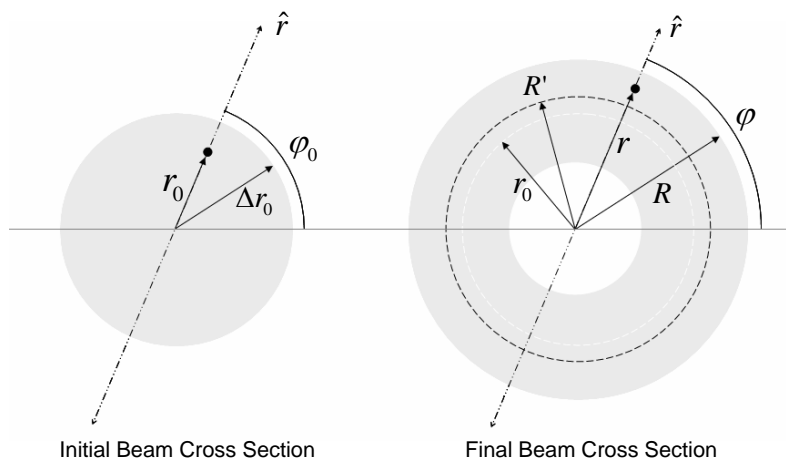


Figure 7.3: This shows graphically what some of the parameters are in find a constraint on the precessionless inhomogeneous Stern-Gerlach effect.

In the semi-classical picture we can describe this as a particle located at r_0 following a classical trajectory with a momentum $p_{r_0} \pm \Delta p_{r_0}$ somewhere within the region of Δr_0 . This is perhaps reminiscent of the pilot-wave picture in which a particle is said to dynamically evolve in unknown ways inside some region of likelihood.

Ignoring the fields then for a moment we can find the amount of radial displacement R' a particle will undergo due to diffraction of the collimator in the same

way that we found the dynamical displacement R . Assuming we can add the *maximal* amount of uncertainty to an initial radial momentum of $p_{r_0} = 0$, i.e. initially no spreading, we have

$$p_{r_0} + \Delta p_{r_0} = \Delta p_{r_0} = mv' = m \left(\frac{R'}{\tau} \right) \quad (7.107)$$

but with a maximum uncertainty state we also have

$$\Delta p_{r_0} \approx \frac{\hbar}{2\Delta r_0} \quad (7.108)$$

so

$$R' \approx \frac{\hbar\tau}{2m\Delta r_0}. \quad (7.109)$$

We interpret R' to be the maximum amount of spreading that will occur due only to the collimation of a given beam of width Δr_0 . We can see that if we make the time of flight τ longer or if we made the beam more narrow, i.e. smaller Δr_0 , the spreading would be more pronounced.

Now, in order to assure ourselves that the spreading from the spin interactions is observable against the inevitably spreading from collimation we require that

$$R > R'. \quad (7.110)$$

Without this condition observation of the ISGE would be swallowed up in a blur.

Using our definitions for R and R' from eq. (41) and eq. (7.109) we get the inequality

$$b\Delta r_0 T > \frac{m}{e} \quad (7.111)$$

where all parameters that can be experimentally adjusted have been consolidated to the left side. Only physical constants are on the right. The semi-classical description tells us that this is the condition for an observable effect.

We have already discussed the field locally in the previous section. At \mathbf{r}_0 there is apparently then a local SGE with a precession frequency of

$$\omega = \frac{e|\mathbf{B}|}{2m} = \frac{e b r_0}{2m} \quad (7.112)$$

about the local field direction \hat{r} .

This could average away the transverse deflections due to the transverse components of the spin if it is allowed to operate long enough. Avoiding this gives us one more experimental condition. We require

$$T \approx \frac{2\pi}{\omega} = \frac{4\pi m}{e b r_0} \quad (7.113)$$

where T is the time of interaction or the time the particle spends in the field. Putting the value of T into the condition of eq. (7.111) we arrive at

$$r_0 < 4\pi \Delta r_0. \quad (7.114)$$

If this condition is met then (1) precession is not a valid argument to discount the measurement of transverse components in the field because it is too slow to sufficiently average them away and yet (2) the spreading from spin effects will be observable despite the spreading due to collimation. If we recall our interpretation of r_0 as the initial location of a particle *within* a beam spot of width Δr_0 it is seen that this condition is *always* met.

This is an extremely interesting result but it must be qualified. (1) It follows an approach that neglects the kinetic energy terms which amounts to dropping important derivatives from a differential equation in \mathbf{x} -space. (2) It is a semi-classical derivation so it is neither purely classical nor purely quantum. Its interpretation stands therefore on unclear ground. Whatever concepts might be most useful from either of the two regimes can be borrowed at will despite their mutual inconsistencies. (3) From (2) all variable definitions are vague. (4) As an example of (3) the momentum p_r was used in one case as a definite value whereas in another case it was said to uphold the uncertainty principle. Therefore, perhaps the inconsistency of the result with the uncertainty principle is a function of the application of the semi-classical representation and not of the phenomenon itself. (5) The \mathbf{B} -field is only approximate. (6) In conjunction with (1) the incident waves were assumed to be infinite plane waves as stated in sections 6.2.1 so in actuality, even if all spin “up” components go “out” and all “down” components go “in” nothing but an infinite blur will be detected.

There are many other concerns that might be mentioned. These are sufficient though to point out that this discussion provides an interesting test, namely the method of this section, that might be applied under any circumstance using any approach or interpretation. Even if physically wrong it is useful in teaching us about our own misunderstandings of both interpretations of the representations and of the representations themselves.

7.11 Other Representations

Amongst the representations explored in this chapter there are several other possible representations by which we can view and assess the question of the ISGE and precession in the standard SGE. We mention here a few of those that may be beneficial.

7.11.1 Propagators

The Green's function method outlined in section 6.9 maps source points in space to field points in space. But this glossed over the time evolution of the system. We have yet to do a solution that does not trivialize the time-dependence but allows it to naturally evolve.⁶ Propagators can do this. They are similar to Green's functions but they map temporally past points to temporally present or future points.

[28] gives analytic expressions for the states $\psi_{\uparrow\downarrow}$ in the coordinates (x, y, z) which supposedly justifies its claim in section 6.12.2. However, as many of the technical details in its derivation are cited from other papers and have not been sufficiently verified or interpreted we only mention this for completeness and to point out the potential fruitfulness of the propagator method of [28] for describing the time evolution in the ISGE.

⁶Section 6.10 is the closest we have come to this so far. Eq. (7.84) might be generalized from assuming incident plane waves ($v_j = 0$) to assuming some sort of incident packet (a spread in v_j).

7.11.2 Numerical Methods

To this point no numerical investigation has been done. While representing these problems numerically gives concrete and definite results they do not offer the same sort of insight as does the explicit confrontation of detail that an analytic approach offers. However, for a computer the PDEs that we have dealt with here could be solved for specific choices of parameters. This would not only give us one more perspective on the phenomenon of the ISGE but one that focused on *results*. This could guide our work in other more revealing analytic approaches including the ones discussed in this chapter. For example, it could provide an estimate for how accurate eq. (7.46) is for the near-origin approximation of section . 7.5.7. For this reason numerical methods should be pursued in the future.

7.11.3 Perturbation Theory

Often a portion of a system can be designated as small. Using perturbation methods these small effects can be eventually accounted for. This generally yields approximate but often accurate results. Although one of the defining characteristics of the ISGE is the *equal* treatment of field components there are several other perturbation techniques that might be applied to solving the ISGE as was the Born method in section 6.9.2.

7.11.4 Bohmian Mechanics

Because we desire to understand the inner workings of the measurement process it is difficult to apply standard quantum techniques to gain understanding. In a field that is formed by measurement axioms⁷ how does one objectively study the nature of measurement itself? The methods of quantum mechanics have been effectively designed and interpreted to match experimental results or potential *results*, not *processes*.

There has been relatively little done with the other approaches that claim to give an accounting of the behind-the-scenes dynamics of the measurement process

⁷Note the discussion of the “collapse” and “cut” axioms in section 3.2.2.

but they do exist. Many have applied the mechanics first introduced by de Broglie in 1927 and formalized by David Bohm in 1952 to inquiries of the SGE. In such cases the actual dynamics of both the position and spin orientation of the particles can be continuously and consistently described [37], [38], [13], [12]. This has even been done using the representational algebra of section 7.7 to account for relativistic effects [39].

Insightful investigations of the sort proposed by Bohm should be extended to the case of the ISGE. Like many other representations they have the potential to offer unique insights into the operation of measurement and the SGE in general.

Chapter 8

Conclusions

There has been an overarching tension in all that we have done in this thesis. It arises from the fact that while representations are necessary in order for rational communication and comprehension they also necessarily alter the perceived behavior of the phenomena they represent. They carry an accompanying value system and set of assumptions. In the case of the traditional SGE these values and assumptions have not been explicitly identified in the past because of its axiomatic role in the modern interpretation and practice of quantum mechanics. In Chapter 5 we attempted to show how this has limited our understanding, or possibility for understanding, in various ways and proposed a study of the ISGE. The question then becomes, “How does one study a phenomenon independent of its representation?” In this thesis we have used several different techniques in an effort to understand the SGE, and more particularly the ISGE, on a deeper level.

8.1 The Method of Relativity

Despite the multiplicity of specific methods that we have used our overall approach is not that different from the general methodology of Einsteinian relativity. Instead of being confused by the relative nature of the several possible points of view that can be taken to solve a single problem we have deliberately moved between these “frames of reference” in order to study nature on a level independent of the frames themselves. That is, by honestly recognizing our *relative* knowledge we can approach more *absolute* knowledge.

In more concrete language we have approached the ISGE using several different methods. Each method necessarily involved some simplifying assumptions and approximations. By comparing the effects of these simplifications, i.e. the results, between several methods we were able to conclude as to which behaviors belong to the method and which belong to the phenomenon. By comparing several techniques we gained confidence in our conclusions.

8.2 Our Results

For example, in our thematic account two distinct momenta were found using the rest frame of the particles in the beam. We noted how this amounts to the assumption of incoming plane waves. This assumption was also made in other sections with similar results. However, using the Heisenberg representation made it more explicit so that one could perhaps lift the assumption in graded steps thus more closely returning to exact solutions.

In our use of differential equations we did not make the assumption of incoming plane waves. It did however show us the effect of not having an asymptotically finite field. Using the integral equations provided by Green's functions provided for both these issues: we could address the asymptotic behavior of the field without having to assume incoming plane waves.

The Heisenberg representation also allowed us to treat the time-dependence in a more natural way and using the Clifford algebra \mathcal{Cl}_3 we found a result that which seemed to emphasize the rotations of a local precession and not the discreteness in the momenta. \mathcal{Cl}_3 also gave us a way to convert systems of coupled equations into a single equations in which the separation of variables might be useful.

Some of our most significant results have had to do precession because it is so vital to the standard description of the SGE. We saw how the effect of precession can be explicitly represented as a clearly mathematical trick without any necessary recourse to the physical picture of rotating spin vectors (section 7.1.3). This doesn't help our intuition but by abstracting away from physical pictures it allows us a view of its mathematical structure. Again, with this knowledge we could reconstruct and

study the effects of precession, gradually moving from the ISGE to the SGE, by keeping higher and higher order terms in the expansion eq. (7.7).

We also saw how it was precession that by and large justified the usual interpretation of the uncertainty principle in Stern-Gerlach measurement by averaging away incompatible components. However, we saw that the precessionless ISGE could require us to use a more definite notion of uncertainty as the appearance of *any localized* spot on the detecting plate seems to violate its usual interpretation (see section 5.5.3).

Thus, the ISGE, which appears to be only a *local* version of the traditional SGE in which the spin separations occur along a *radial* axis as defined by the initial position of the particle, has opened the door for a clearer study of both the theory and interpretation of physical science. We believe there are still several interesting and unanswered questions as to the correct interpretation of the SGE. Being very subtle and “relative” issues these may be pursued using the methodology of relativity as we have used it here.

8.3 The Dangers of an Inadequate Philosophy

Perhaps one of the most interesting conclusions of applying methods involving several representations is not the absolute knowledge that comes out but that the apparent fact that there *is* absolute knowledge. For this reason we cannot be too content with the prevalent “it works” attitude although it may be necessary for the purposes of instruction. That is, “it works” should only serve as a temporary justification for pursuing knowledge and not as a permanent replacement. Whether or not something “works,” which is a relative term, theoretical and analytical studies should be encouraged - *using many different representations* - as only they provide explicit conceptual confrontation with all the details of nature. This will allow us to form a more accurate picture of physical processes, for, as we hinted at in section 2.1

Without abstract ideas...‘you would not be able to deal with concrete, particular, real-life problems. You would be in the position of a new-born infant, to whom every object is a unique, unprecedented phenomenon...As

a human being [we] have no choice about the fact that [we] need a philosophy. [Our] only choice is whether [we] define [our] philosophy by a conscious, rational, disciplined process of thought and scrupulously logical deliberation - or let [our] subconscious accumulate a junk heap of unwarranted conclusions, false generalizations, undefined contradictions, undigested slogans, unidentified wishes, doubts and fears, thrown together by chance, but integrated by [our] subconscious into a kind of mongrel philosophy and fused into a single, solid weight: self-doubt, like a ball and chain in the place where [our] minds's wings should have grown. [3] (p. 1-2)

In summary, the search for a more complete and consistent understanding of the SGE has led us to an understanding, not only of the SGE and precession, but more generally of the process of representation. We have seen that in the act of representing a phenomenon for communication or study there is an inherent tension contributes can contribute to our progress. In research this is manifest as the tension between accurate historical accounts and clearly formed thematic ones, between the physical and analytical justifications of our approximations, between the the realist and positivist interpretations of physics, between the \mathbf{x} and \mathbf{p} representations, between mutually inconsistent approximation methods, etc. In physics education and teaching this tension lies in the necessary balance of both clear and accurate communication. In any case we have seen the value of not only understanding the disparities of our representations but also in accepting and using them for our advantage and progress. Thus, because of the inadequacy and necessity of our representations, progress-oriented research and teaching require that we master the art of mediation.

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